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hep-ph/9907494SCALAR ONE-LOOP INTEGRALS USING THE  
NEGATIVE-DIMENSION APPROACHC. Anastasiou<sup>1</sup>, E. W. N. Glover<sup>2</sup> and C. Oleari<sup>3</sup>*Department of Physics, University of Durham, Durham DH1 3LE, England***Abstract**

We study massive one-loop integrals by analytically continuing the Feynman integral to negative dimensions as advocated by Halliday and Ricotta and developed by Suzuki and Schmidt. We consider  $n$ -point one-loop integrals with arbitrary powers of propagators in general dimension  $D$ . For integrals with  $m$  mass scales and  $q$  external momentum scales, we construct a template solution valid for all  $n$  which allows us to obtain a representation of the graph in terms of a finite sum of generalised hypergeometric functions with  $m+q-1$  variables. All solutions for all possible kinematic regions are given simultaneously, allowing the investigation of different ranges of variation of mass and momentum scales.

As a first step, we develop the general framework and apply it to massive bubble and vertex integrals. Of course many of these integrals are well known and we show that the known results are recovered. To give a concrete new result, we present expressions for the general vertex integral with one off-shell leg and two internal masses in terms of hypergeometric functions of two variables that converge in the appropriate kinematic regions. The kinematic singularity structure of this graph is sufficiently complex to give insight into how the negative-dimension method operates and gives some hope that more complicated graphs can also be evaluated.

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# 1 Introduction

Loop integrals play an important role in making precise perturbative predictions in quantum field theory, in general, and in the Standard Model of particle physics, in particular. As such, a large effort has been expended in developing methods for evaluating them. The problem is complicated by the appearance of ultraviolet (UV) and infrared (IR) singularities, and it has become customary to use dimensional regularisation [1, 2, 3] to extend the dimensionality of the loop integral away from 4-dimensions to  $D = 4 - 2\epsilon$ , to regulate the infrared and ultraviolet singularities.

With the increasing of the number of legs, of the number of mass scales or of the number of loops, the integrals can be made almost arbitrarily complex and difficult to solve analytically. Different methods [4]–[11] have been developed to solve the Feynman integrals. We mention here only two of them: the integration by parts [7], which works well for some two-loop vertex diagrams [12] reducing them to simpler known graphs with different powers of the propagators (however this breaks down for more complicated graphs such as the double box, where irreducible numerators factors are present) and the Mellin-Barnes integral representation (see for example [13]), which was successfully used by Smirnov [14] to calculate the two-loop box integral. In this approach, the integral is usually written as multiple contour integrals of  $\Gamma$  functions and powers of ratios of the mass scales in the problem. By closing the contour, we obtain an infinite series of residues at the singular points of the  $\Gamma$  functions. These series can be identified as generalised hypergeometric functions, whose convergence properties reflect the threshold-singularity structure of the integral.

There are several advantages in using hypergeometric functions to represent the integral. First, these hypergeometric functions often have integral representations themselves, in which an expansion in  $\epsilon$  can be made, yielding expressions in logarithms, dilogarithms etc.. It seems that, where direct evaluation of the hypergeometric function in terms of known functions is possible, very compact results are obtained [15, 16]. Second, because the series is convergent and well behaved in a particular region of phase space, it can be numerically evaluated [17]. In fact, each hypergeometric representation immediately allows an asymptotic expansion of the integral in terms of ratios of momentum and mass scales. Third, through analytic continuation formulae, the hypergeometric functions valid in one kinematic domain can be re-expressed in a different kinematic region.

Not all work has concentrated around  $D = 4$ . In fact, a close connection between tensor loop integrals - those with additional powers of the loop momentum in the numerator - and higher-dimension scalar integrals ( $D = 6 - 2\epsilon$ , for example) is well established [11, 18, 19, 20]. Furthermore, in 1987, Halliday and Ricotta [21, 22] suggested that it would be useful to calculate the loop integral considering  $D$  as a negative number. Because loop integrals are analytic in the number of dimensions  $D$  (and also in the powers of the propagators) they

proposed to calculate the integral in negative dimensions and return to positive dimensions, and specifically  $D = 4 - 2\epsilon$ , after the integrations have been performed. As we will discuss more fully later on, integration over the loop momentum and/or the parameters introduced to do the loop integration is replaced with infinite series, which again can be identified as generalised hypergeometric functions. Recently this idea has been picked up again by Suzuki and Schmidt who have evaluated a number of two-loop integrals [23], three-loop integrals [24], one-loop tensor integrals [25] as well as the one-loop massive box integral for the scattering of light by light [26]. In this latter case, as well as reproducing the known hypergeometric-series representations of Ref. [15], valid in particular kinematic regions, Suzuki and Schmidt simultaneously found hypergeometric solutions valid in other kinematic domains. Of course, all of these solutions are related by analytic continuation. However, it is easy to envisage integrals that yield hypergeometric functions where the analytic continuation formulae are not known a priori. In these cases, having series expansions directly available in all kinematic regions is useful.

In this paper we wish to explore the negative-dimension approach (NDIM) further. In particular we focus on one-loop integrals with general powers of the propagators and arbitrary dimension  $D$ . There are several reasons for doing this. First, it allows connection with the general tensor-reduction program based on integration by parts of Refs. [19, 20]. Here the tensor integrals are linear combinations of scalar integrals with either higher dimension or propagators raised to higher powers. Second, we can imagine inserting the one-loop results into a two-loop integral by closing up external legs. This is trivial for most bubble integrals, but more complicated for vertex and box graphs. Broadhurst [27] has shown that this is possible for the non-trivial two-loop self-energy graph. Third, it actually simplifies the calculation. As we will show, by keeping the parameters general, it is easier to identify the regions of convergence of the hypergeometric series and therefore which hypergeometric functions to group together. For specific values of the parameters, the hypergeometric functions often collapse to simpler functions. As a first step, we develop the general framework and apply it to massive bubble and vertex integrals. Of course many of these results are well known. However, they serve to iron out some of the subtleties of the NDIM approach. To give a concrete new result, we present expressions for the vertex integral with one off-shell leg and two internal masses in arbitrary dimension and for general powers of the propagators in terms of hypergeometric functions of two variables. The kinematic singularity structure of this graph is sufficiently complex to give insight into how NDIM operates and gives some hope that more complicated graphs can also be evaluated.

Our paper is organised as follows. In Sec. 2.2 we first review the theoretical framework of one-loop integrals with Schwinger parameters and briefly explain the basic idea of integrating in negative dimensions. We then apply NDIM to construct template solutions for arbitrary one-loop integrals together with a linear system of constraints that relates the powers of the propagators in the loop integral to the summation variables. The system of constraints has

many solutions and each one must be inserted into the template solution, yielding a sum over fewer variables that can be identified as a generalised hypergeometric function. This method gives simultaneously all the solutions in all the possible kinematic regions. The approach is illustrated for the massive bubble integral where we show how to recover the known results. We discuss how the form of the solution in different kinematical regimes is dictated by the convergence properties of the hypergeometric functions and the structure of the system. In Sec. 3 we consider one-loop triangles and give the form of the template solution and the system of constraints with arbitrary powers of the propagators, internal masses, external legs off-shell and for general  $D$ . We apply this result to triangle integrals with three scales and give expressions valid in the various kinematic regions appropriate to the vertex integral. Results are given in the form of hypergeometric functions of one and two variables, which are defined in Appendix A. For specific choices of  $D$  and the propagator powers, these functions can be evaluated as logarithms and dilogarithms using the integral representations that are also provided in the appendix together with a list of hypergeometric identities that often simplify the results. Finally, our method is summarized in Sec. 4.

## 2 Theoretical framework

The generic  $n$ -point one-loop integral in  $D$ -dimensional Minkowski space with loop momentum  $k$  is given by

$$I_n^D(\{\nu_i\}; \{Q_i^2\}, \{M_i^2\}) = \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{A_1^{\nu_1} \dots A_n^{\nu_n}}, \quad (2.1)$$

where, as indicated in Fig. 1, the external momenta  $k_i$  are all incoming so that  $\sum_{i=1}^n k_i^\mu = 0$  and the propagators have the form

$$\begin{aligned} A_1 &= k^2 - M_1^2 + i0, \\ A_i &= \left( k + \sum_{j=1}^{i-1} k_j \right)^2 - M_i^2 + i0 \quad i \neq 1, \end{aligned} \quad (2.2)$$

$M_i$  being the mass of the  $i$ th propagator. The external momentum scales are indicated with  $\{Q_i^2\}$ . For standard integrals, the powers  $\nu_i$  to which each propagator is raised are usually unity. However, we wish, where possible, to leave the powers as general as possible. As discussed earlier, this may have some advantages in evaluating two-loop integrals where often one-loop integrals with arbitrary powers can be inserted into the second loop integration.

To evaluate this integral, we introduce a Schwinger parameter  $x_i$  for each propagator (noting that  $A_i < 0$  after Wick rotation to Euclidean space) so that

$$\frac{1}{A_i^{\nu_i}} = \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty dx_i x_i^{\nu_i-1} \exp(x_i A_i), \quad (2.3)$$

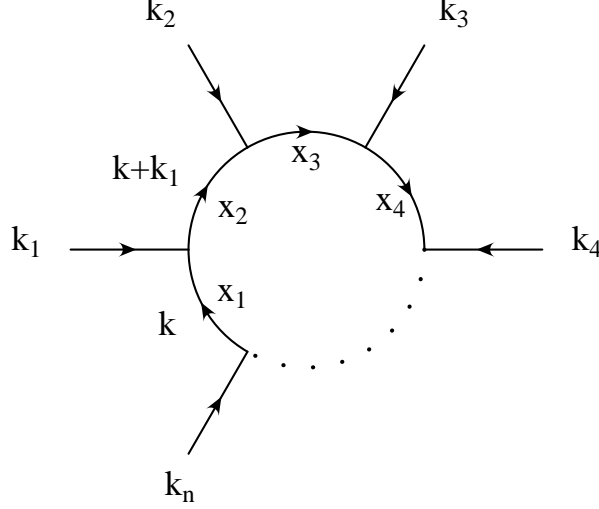


Figure 1: The generic one-loop graph

and we can rewrite Eq. (2.1) as

$$I_n^D \left( \{\nu_i\}; \{Q_i^2\}, \{M_i^2\} \right) = \int \mathcal{D}x \int \frac{d^D k}{i\pi^{D/2}} \exp \left( \sum_{i=1}^n x_i A_i \right), \quad (2.4)$$

where we have used the shorthand

$$\int \mathcal{D}x = (-1)^\sigma \left( \prod_{i=1}^n \frac{1}{\Gamma(\nu_i)} \int_0^\infty dx_i x_i^{\nu_i-1} \right), \quad (2.5)$$

with

$$\sigma = \sum_{i=1}^n \nu_i. \quad (2.6)$$

The Gaussian integral over the loop momentum can be solved in a straightforward way, and using the Minkowski space relation

$$\int \frac{d^D k}{i\pi^{D/2}} \exp(\alpha k^2) = \frac{1}{\alpha^{D/2}}, \quad (2.7)$$

we have the usual Minkowski space result

$$I_n^D \left( \{\nu_i\}; \{Q_i^2\}, \{M_i^2\} \right) = \int \mathcal{D}x \frac{1}{\mathcal{P}^{D/2}} \exp(\mathcal{Q}/\mathcal{P}) \exp(-\mathcal{M}). \quad (2.8)$$

The quantities  $\mathcal{P}$  and  $\mathcal{M}$  are given by

$$\mathcal{P} = \sum_{i=1}^n x_i, \quad (2.9)$$

$$\mathcal{M} = \sum_{i=1}^n x_i M_i^2, \quad (2.10)$$

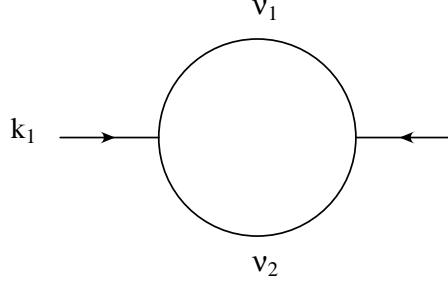


Figure 2: The one-loop vacuum graph

while  $\mathcal{Q}$  may be simply read off from the Feynman diagram

$$\mathcal{Q} = \sum_{i=1}^{n-1} \sum_{j=i+1}^n x_i x_j \left( \sum_{k=i}^{j-1} k_k \right)^2 = \sum_{i=1}^q \mathcal{Q}_i. \quad (2.11)$$

Each of the  $q$  terms in  $\mathcal{Q}$  is indicated with  $\mathcal{Q}_i$ , and is obtained by cutting the loop diagram into two across propagators  $a$  and  $b$  and constructing the four-momentum  $Q_i^\mu$  on each side of the cut:  $\mathcal{Q}_i = x_a x_b Q_i^2$ . For example, the one-loop bubble graph shown in Fig. 2 has two propagators ( $n = 2$ ), so that  $\mathcal{P} = x_1 + x_2$  and  $\mathcal{M} = x_1 M_1^2 + x_2 M_2^2$ .  $\mathcal{Q}$  is obtained by examining the momentum flowing across the only possible cut ( $q = 1$ ):  $\mathcal{Q} = \mathcal{Q}_1 = x_1 x_2 Q_1^2$ , with  $Q_1^2 = k_1^2$ .

## 2.1 The negative-dimension approach

The crucial point in the negative-dimension approach is that the Gaussian integral (2.7) is an analytic function of the space-time dimension. Hence it is possible to consider  $D < 0$  and to make the definition [21]

$$\int \frac{d^D k}{i\pi^{D/2}} (k^2)^n = n! \delta_{n+\frac{D}{2}, 0} \quad (2.12)$$

for positive values of  $n$ . We see that by expanding the exponential in (2.7) and inserting the definition (2.12), after the exchange of the integration with respect to the summation

$$\int \frac{d^D k}{i\pi^{D/2}} \exp(\alpha k^2) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int \frac{d^D k}{i\pi^{D/2}} (k^2)^n = \frac{1}{\alpha^{D/2}}, \quad (2.13)$$

we recover the original result, provided that  $D$  is both negative and even (so that the Kronecker  $\delta$  can be satisfied and the contribution with  $n = -D/2$  selected from the sum). We note that, with this definition, negative-dimensional integrals can be shown to obey the necessary translation properties [21].

## 2.2 The general case: different masses

For the one-loop integrals we are interested in here, we follow the approach suggested by Suzuki and Schmidt [23]–[26] and view Eqs. (2.4) and (2.8) as existing in negative dimensions. Making the same series expansion of the exponential as above, Eq. (2.4) becomes

$$\begin{aligned} I_n^D(\{\nu_i\}; \{Q_i^2\}, \{M_i^2\}) &= \int \mathcal{D}x \sum_{n_1, \dots, n_n=0}^{\infty} \int \frac{d^D k}{i\pi^{D/2}} \prod_{i=1}^n \frac{(x_i A_i)^{n_i}}{n_i!} \\ &= \int \mathcal{D}x \sum_{n_1, \dots, n_n=0}^{\infty} I_n^D(-n_1, \dots, -n_n; \{Q_i^2\}, \{M_i^2\}) \prod_{i=1}^n \frac{x_i^{n_i}}{n_i!}, \end{aligned} \quad (2.14)$$

where the  $n_i$  are positive integers. The target loop integral is an infinite sum of (integrals over the Schwinger parameters of) loop integrals with negative powers of the propagators.

Likewise, we expand the exponentials in Eq. (2.8)

$$I_n^D(\{\nu_i\}; \{Q_i^2\}, \{M_i^2\}) = \int \mathcal{D}x \sum_{n=0}^{\infty} \frac{\mathcal{Q}^n \mathcal{P}^{-n-\frac{D}{2}}}{n!} \sum_{m=0}^{\infty} \frac{(-\mathcal{M})^m}{m!}, \quad (2.15)$$

and introduce the integers  $q_1, \dots, q_q$ ,  $p_1, \dots, p_n$  and  $m_1, \dots, m_n$  to make multinomial expansions of  $\mathcal{Q}$ ,  $\mathcal{P}$  and  $\mathcal{M}$  respectively

$$\begin{aligned} \mathcal{Q}^n &= \sum_{q_1, \dots, q_q=0}^{\infty} \frac{\mathcal{Q}_1^{q_1}}{q_1!} \dots \frac{\mathcal{Q}_q^{q_q}}{q_q!} (q_1 + \dots + q_q)! \\ \mathcal{P}^{-n-\frac{D}{2}} &= \sum_{p_1, \dots, p_n=0}^{\infty} \frac{x_1^{p_1}}{p_1!} \dots \frac{x_n^{p_n}}{p_n!} (p_1 + \dots + p_n)! \\ (-\mathcal{M})^m &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(-x_1 M_1^2)^{m_1}}{m_1!} \dots \frac{(-x_n M_n^2)^{m_n}}{m_n!} (m_1 + \dots + m_n)!, \end{aligned} \quad (2.16)$$

subject to the constraints

$$\sum_{i=1}^q q_i = n, \quad \sum_{i=1}^n p_i = -n - \frac{D}{2} \quad \text{and} \quad \sum_{i=1}^n m_i = m. \quad (2.17)$$

Altogether, Eqs. (2.15) and (2.16) give

$$\begin{aligned} I_n^D(\{\nu_i\}; \{Q_i^2\}, \{M_i^2\}) &= \\ &= \int \mathcal{D}x \sum_{\substack{p_1, \dots, p_n=0 \\ q_1, \dots, q_q=0 \\ m_1, \dots, m_n=0}}^{\infty} \frac{\mathcal{Q}_1^{q_1} \dots \mathcal{Q}_q^{q_q}}{q_1! \dots q_q!} \frac{x_1^{p_1} \dots x_n^{p_n}}{p_1! \dots p_n!} \frac{(-x_1 M_1^2)^{m_1}}{m_1!} \dots \frac{(-x_n M_n^2)^{m_n}}{m_n!} (p_1 + \dots + p_n)!, \end{aligned} \quad (2.18)$$

with the constraints expressed by Eq. (2.17).



We recall that each of the  $\mathcal{Q}_i$  is a bilinear in the Schwinger parameters, so that the target loop integral is now an infinite sum of powers of the scales of the process (with each of the  $M_i^2$  and the  $Q_i^2$  raised to a different summation variable) integrated over the Schwinger parameters.

Equations (2.14) and (2.18) are two different expressions for the same quantity:  $I_n^D$ . However, rather than performing the integrals over the  $x_i$ 's, we use the fact that the  $x_i$ 's are independent parameters, so that the integrands themselves must be equivalent:

$$\sum_{n_1, \dots, n_n=0}^{\infty} I_n^D \left( -n_1, \dots, -n_n; \{Q_i^2\}, \{M_i^2\} \right) \prod_{i=1}^n \frac{x_i^{n_i}}{n_i!} = \sum_{\substack{p_1, \dots, p_n=0 \\ q_1, \dots, q_q=0 \\ m_1, \dots, m_n=0}}^{\infty} \frac{\mathcal{Q}_1^{q_1} \dots \mathcal{Q}_q^{q_q}}{q_1! \dots q_q!} \frac{x_1^{p_1} \dots x_n^{p_n}}{p_1! \dots p_n!} \frac{(-x_1 M_1^2)^{m_1}}{m_1!} \dots \frac{(-x_n M_n^2)^{m_n}}{m_n!} (p_1 + \dots + p_n)! \quad (2.19)$$

Comparing term by term the left-hand-side (lhs) with the right-hand-side (rhs) allows us to read off the value of  $I_n^D$ . In fact, the coefficient of the term  $x_1^{-\nu_1} \dots x_n^{-\nu_n}$  in the lhs of Eq. (2.19), where the  $\nu_i$  are negative integers, is given by

$$I_2^D(\nu_1, \dots, \nu_n) \left( \prod_{i=1}^n \frac{1}{\Gamma(1 - \nu_i)} \right). \quad (2.20)$$

This term is equal to the coefficient of the term  $x_1^{-\nu_1} \dots x_n^{-\nu_n}$  in the rhs of Eq. (2.19). Writing a general expression is not possible, since the  $\mathcal{Q}_i$  are process dependent. Nevertheless, we can extract the momentum scale  $Q_i^2$  from each of the  $\mathcal{Q}_i$  and find the coefficient of this term to be

$$\sum_{\substack{p_1, \dots, p_n=0 \\ q_1, \dots, q_q=0 \\ m_1, \dots, m_n=0}}^{\infty} (Q_1^2)^{q_1} \dots (Q_q^2)^{q_q} (-M_1^2)^{m_1} \dots (-M_n^2)^{m_n} \times \left( \prod_{i=1}^n \frac{1}{\Gamma(1 + m_i) \Gamma(1 + p_i)} \right) \left( \prod_{i=1}^q \frac{1}{\Gamma(1 + q_i)} \right) \Gamma \left( 1 + \sum_{k=1}^n p_k \right), \quad (2.21)$$

subject to the  $n$  (process-dependent) constraints that ensure that the power of  $x_i$  on the lhs ( $-\nu_i$ ) is equal to the power of  $x_i$  on the rhs, which is generally a combination of integers of the summation.

By adding together the first two expressions in Eq. (2.17), we obtain an additional constraint, that is

$$p_1 + \dots + p_n + q_1 + \dots + q_q = -\frac{D}{2}. \quad (2.22)$$

Equating Eqs. (2.20) and (2.21), we obtain an expression for the loop integral with

negative powers of the propagators in negative dimensions

$$\begin{aligned}
I_n^D(\{\nu_i\}; \{Q_i^2\}, \{M_i^2\}) &\equiv \sum_{\substack{p_1, \dots, p_n=0 \\ q_1, \dots, q_q=0 \\ m_1, \dots, m_n=0}}^{\infty} (Q_1^2)^{q_1} \dots (Q_q^2)^{q_q} (-M_1^2)^{m_1} \dots (-M_n^2)^{m_n} \\
&\times \left( \prod_{i=1}^n \frac{\Gamma(1 - \nu_i)}{\Gamma(1 + m_i)\Gamma(1 + p_i)} \right) \left( \prod_{i=1}^q \frac{1}{\Gamma(1 + q_i)} \right) \Gamma\left(1 + \sum_{k=1}^n p_k\right).
\end{aligned} \tag{2.23}$$

Equation (2.23), together with the constraints, is the main result of this paper. The loop integral is written directly as an infinite sum. Given that  $\mathcal{Q}$  can be read off directly from the Feynman graph, so can the precise form of Eq. (2.23) as well as the system of constraints. Of course, strictly speaking we have assumed that both  $\nu_i$  and  $D/2$  are negative integers and we must be careful in interpreting this result in the physically interesting domain where the  $\nu_i$  and  $D$  are all positive. However, this is relatively straightforward and in the following sections we show how quite general results for one-loop massive bubbles and triangles can be obtained.

**Example:** to give an explicit example of how Eq. (2.23) and the system of constraints appear, we consider the one-loop bubble with different masses. Equations (2.14) and (2.18) become

$$\begin{aligned}
&I_2^D(\nu_1, \nu_2; Q_1^2, M_1^2, M_2^2) \\
&= \int \mathcal{D}x \sum_{n_1, n_2=0}^{\infty} I_2^D(-n_1, -n_2; Q_1^2, M_1^2, M_2^2) \frac{x_1^{n_1} x_2^{n_2}}{n_1! n_2!} \\
&= \int \mathcal{D}x \sum_{p_1, p_2, q_1, m_1, m_2=0}^{\infty} \frac{(x_1 x_2 Q_1^2)^{q_1} x_1^{p_1} x_2^{p_2} (-x_1 M_1^2)^{m_1} (-x_2 M_2^2)^{m_2}}{q_1! p_1! p_2! m_1! m_2!} (p_1 + p_2)!,
\end{aligned} \tag{2.24}$$

so that, by selecting powers of  $x_1^{-\nu_1}$  and  $x_2^{-\nu_2}$ , we find (see Eq. (2.23))

$$\begin{aligned}
I_2^D(\nu_1, \nu_2; Q_1^2, M_1^2, M_2^2) &= \sum_{p_1, p_2, q_1, m_1, m_2=0}^{\infty} (Q_1^2)^{q_1} (-M_1^2)^{m_1} (-M_2^2)^{m_2} \\
&\times \frac{\Gamma(1 - \nu_1)\Gamma(1 - \nu_2)\Gamma(1 + p_1 + p_2)}{\Gamma(1 + m_1)\Gamma(1 + m_2)\Gamma(1 + p_1)\Gamma(1 + p_2)\Gamma(1 + q_1)},
\end{aligned} \tag{2.25}$$

together with the system of constraints

$$\begin{aligned}
q_1 + p_1 + m_1 &= -\nu_1, \\
q_1 + p_2 + m_2 &= -\nu_2, \\
q_1 + p_1 + p_2 &= -\frac{D}{2}.
\end{aligned} \tag{2.26}$$

In Sec. 2.4, we will show how this particular system can be solved to give results for the bubble integral in positive dimensions  $D$ , with arbitrary positive powers of the propagators.

## 2.3 The general form of the solutions

In general, for an  $n$ -point one-loop integral with  $q$  external momentum scales and  $m$  mass scales, there will be  $(n + q + m)$  summation variables and  $(n + 1)$  constraints. Altogether we expect  $(n + q + m)!/(n + 1)!(q + m - 1)!$  possible solutions (some of which will be eliminated by the specific form of the system of constraints). It is easy to see that these solutions span physically different kinematic regions (depending on the powers of the kinematic scales) and the summations will only converge in the appropriate kinematic domain. We expect that solutions in one kinematic region should be analytically linked to those in other domains.

Each solution of the system of constraints, once inserted into the template of Eq. (2.23), has the following generic form

$$\mathcal{PRE} \times \mathcal{SUM}, \quad (2.27)$$

where we have introduced the following notation:

- $\mathcal{SUM}$  is the sum over the terms that contain unconstrained indices of summation. Instead of dealing with  $\Gamma$  functions, we have formed Pochhammer symbols, defined as

$$(z, n) \equiv \frac{\Gamma(z + n)}{\Gamma(z)}, \quad (2.28)$$

because they are the most suitable way to write generalized hypergeometric functions. For example, in the case where there is only one remaining summation variable  $n$ , then  $\mathcal{SUM}$  takes the form

$$\mathcal{SUM} \sim \sum_{n=0}^{\infty} \frac{(a_1, n) \dots (a_N, n)}{(b_1, n) \dots (b_{N-1}, n)} \frac{x^n}{n!}, \quad (2.29)$$

where  $x$  is the ratio of kinematic scales. The variables  $a_i$  and  $b_i$  are linear in the  $\nu_i$  and  $D$  and do not depend on the summation variables. To put  $\mathcal{SUM}$  in this form, it is often convenient to use the identity (see Eq. (2.32))

$$(z, -n) = (-1)^n \frac{1}{(1 - z, n)}. \quad (2.30)$$

In most cases,  $\mathcal{SUM}$  can be directly identified as a generalized hypergeometric function, in the region of convergence of the series. In general, these hypergeometric functions are analytic and may be evaluated at positive values of  $D$  and  $\nu_i$ .

- The prefactor  $\mathcal{PRE}$  contains all the rest of the terms that are not included in  $\mathcal{SUM}$ . More precisely, it is a product of external scales raised to fixed powers, and  $\Gamma$  functions that do not depend on the summation variables. These may be produced either directly from the particular solution of the system, or in the generation of the Pochhammer symbols.

In the general case of an  $n$ -point one-loop integral with  $q$  external momentum scales and  $m$  mass scales, inspection of Eq. (2.23) dictates that we produce:

- $n$   $\Gamma$  functions with argument  $(1 - \nu_i)$ ,
- $(n + m + q)$  factorials of the summation variables in the denominator,
- one  $\Gamma$  function in the numerator:  $\Gamma(1 + \sum_{k=1}^n p_k)$ .

Applying the  $(n + 1)$  constraints leaves  $(m + q - 1)$  factorials of the remaining unconstrained summation variables and produces an additional  $(n + 1)$   $\Gamma$  functions in the denominator and one in the numerator, as Pochhammer symbols are formed using Eq. (2.28). Altogether there will be  $(n + 2)$  Pochhammer symbols in  $\mathcal{SUM}$ , while  $\mathcal{PRE}$  will be a ratio with  $(n + 1)$   $\Gamma$  functions in *both* numerator and denominator. In both  $\mathcal{SUM}$  and  $\mathcal{PRE}$ , the number of functions may be reduced if there are cancellations between numerator and denominator.

For physical loop integrals with positive powers of propagators, we need to evaluate  $\mathcal{PRE}$  at positive values of the  $\nu_i$  and positive  $D$ . A problem is immediately obvious: the numerator of  $\mathcal{PRE}$  contains  $\Gamma(1 - \nu_i)$ , so that, for positive integer values  $\nu_i$ , it appears that we need to evaluate the  $\Gamma$  functions for negative arguments, where they are singular. However,  $\mathcal{PRE}$  is an analytic function and these singularities cancel between the numerator and denominator.

In fact, it can be easily shown that, starting from the identity

$$\Gamma(z + 1) = z \Gamma(z), \quad (2.31)$$

we have

$$\frac{\Gamma(z)}{\Gamma(z - n)} = (-1)^{-n} \frac{\Gamma(n + 1 - z)}{\Gamma(1 - z)}, \quad (2.32)$$

where  $z$  is a real (or complex) number, and  $n$  is a positive integer.

In the product of  $\Gamma$  functions in the numerator and denominator of the  $\mathcal{PRE}$  term, we can make an iterated use of the identity (2.32), provided we treat  $D/2$  as an integer, as we have already done in the multinomial expansion. We can then rewrite the  $\Gamma$ -function prefactor in a more amenable way by flipping all of the  $\Gamma$  functions from numerator to denominator and vice versa

$$\prod_{i=1}^{n+1} \frac{\Gamma(\alpha_i)}{\Gamma(\beta_i)} = (-1)^{\sum_{i=1}^{n+1} (\beta_i - \alpha_i)} \prod_{i=1}^{n+1} \frac{\Gamma(1 - \beta_i)}{\Gamma(1 - \alpha_i)}, \quad (2.33)$$

where the index  $i$  runs over all  $(n + 1)$   $\Gamma$  functions in the numerator and denominator of  $\mathcal{PRE}$ . In addition, it can be shown that

$$\sum_{i=1}^{n+1} (\beta_i - \alpha_i) = \frac{D}{2}, \quad (2.34)$$

which is independent of the  $\nu_i$ .

### 2.3.1 An example: the massless bubble

Returning to the example of the one-loop self-energy diagram introduced in Sec. 2.2, and setting the masses of the internal lines to zero,  $M_1 = M_2 = 0$  (which is equivalent to terminating the series in  $m_1$  and  $m_2$  at the first term), we obtain the simpler system of constraints (see Eq. (2.26) with  $m_1 = m_2 = 0$ )

$$\begin{aligned} q_1 + p_1 &= -\nu_1, \\ q_1 + p_2 &= -\nu_2, \\ q_1 + p_1 + p_2 &= -\frac{D}{2}. \end{aligned} \quad (2.35)$$

Since  $m = 0$ ,  $q = 1$  and  $n = 2$ , we expect that the  $(n + 1) = 3$  constraints exactly determine the  $(m + q + n) = 3$  variables. In this case  $\mathcal{SUM} = 1$  and the result is entirely given by the prefactor  $\mathcal{PRE}$ . Solving this system yields

$$\begin{aligned} q_1 &= \frac{D}{2} - \nu_1 - \nu_2, \\ p_1 &= \nu_2 - \frac{D}{2}, \\ p_2 &= \nu_1 - \frac{D}{2}. \end{aligned}$$

Inserting these values directly into Eq. (2.25) with  $m_1 = m_2 = 0$  we find

$$\begin{aligned} I_2^D(\nu_1, \nu_2; Q_1^2, 0, 0) &= \mathcal{PRE} \\ &= \frac{\Gamma(1 - \nu_1) \Gamma(1 - \nu_2) \Gamma(1 + \nu_1 + \nu_2 - D)}{\Gamma(1 + \nu_1 - \frac{D}{2}) \Gamma(1 + \nu_2 - \frac{D}{2}) \Gamma(1 + \frac{D}{2} - \nu_1 - \nu_2)} (Q_1^2)^{\frac{D}{2} - \nu_1 - \nu_2}. \end{aligned} \quad (2.36)$$

As expected there are  $(n + 1)$   $\Gamma$  functions in both numerator and denominator and furthermore the arguments satisfy Eq. (2.34). We therefore apply Eq. (2.33) and find

$$I_2^D(\nu_1, \nu_2; Q_1^2, 0, 0) = (-1)^{\frac{D}{2}} \frac{\Gamma(\frac{D}{2} - \nu_1) \Gamma(\frac{D}{2} - \nu_2) \Gamma(\nu_1 + \nu_2 - \frac{D}{2})}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(D - \nu_1 - \nu_2)} (Q_1^2)^{\frac{D}{2} - \nu_1 - \nu_2}, \quad (2.37)$$

where  $\nu_1$  and  $\nu_2$  are positive and which agrees with the known result straightforwardly obtained using Feynman parameters.

## 2.4 Massive bubble integrals

We want now to give a detailed description of how to build the solutions starting from the general form (2.23) for the loop integral and from the system of constraints, and we want to discuss how the solutions of the system of constraints need to be combined to give a meaningful result.

We will refer to a precise example to make things clearer: the bubble integral with different masses in the propagators of Eq. (2.25). Particular cases with  $\nu_1 = \nu_2 = 1$  are important in electroweak renormalization and have been known for some time (see for example [2]). The more general cases with  $\nu_1 \neq \nu_2 \neq 1$  have been studied by Boos and Davydychev [13] using the Mellin-Barnes integral representations.

As discussed at the end of Sec. 2.2, in the case where the two masses are non-zero and different, there are  $(m+n+q) = 5$  summation variables: two for the propagator masses ( $m_1$  and  $m_2$ ), two for the expansion of  $\mathcal{P}$  ( $p_1$  and  $p_2$ ) and one for the external momentum scale ( $q_1$ ).

The  $(n+1) = 3$  constraints are given in Eq. (2.26). There are a maximum of  $5!/3!/2! = 10$  possible solutions, one for each of the ways in which we can choose three variables among the five, and solve the system with respect to these triplets. In this case, there is no solution if we try to solve the system for  $\{p_1, q_1, m_2\}$  or  $\{p_2, q_1, m_1\}$ , so that we have only eight solutions.

Each of the eight solutions corresponds to different values of the integer summation variables and we insert each of them into the general expression for the propagator integral, Eq. (2.25). For example, solving for  $\{p_1, p_2, q_1\}$ , yields

$$\begin{aligned} p_1 &= \nu_2 + m_2 - \frac{D}{2}, \\ p_2 &= \nu_1 + m_1 - \frac{D}{2}, \\ q_1 &= \frac{D}{2} - \nu_1 - \nu_2 - m_1 - m_2, \end{aligned} \tag{2.38}$$

and the contribution of this solution to the integral (2.25) is

$$\begin{aligned} I_2^{\{m_1, m_2\}} &= \sum_{m_1, m_2=0}^{\infty} \left(Q_1^2\right)^{\frac{D}{2}-\nu_1-\nu_2-m_1-m_2} \left(-M_1^2\right)^{m_1} \left(-M_2^2\right)^{m_2} \frac{\Gamma(1-\nu_1)\Gamma(1-\nu_2)}{\Gamma(1+m_1)\Gamma(1+m_2)} \\ &\times \frac{\Gamma(1+\nu_1+\nu_2-D+m_1+m_2)}{\Gamma\left(1+\nu_2-\frac{D}{2}+m_2\right)\Gamma\left(1+\nu_1-\frac{D}{2}+m_1\right)\Gamma\left(1+\frac{D}{2}-\nu_1-\nu_2-m_1-m_2\right)}, \end{aligned} \tag{2.39}$$

where we have labelled the integral with respect to the indices of summation and we have dropped the functional dependence of  $I_2^D$ , for ease of notation.

As discussed in Sec. 2.2, we now form the Pochhammer symbols, and we make use of the Eq. (2.30) to flip the Pochhammer symbol in the denominator with negative indices of summation, to obtain

$$I_2^{\{m_1, m_2\}} = \left(Q_1^2\right)^{\frac{D}{2}-\nu_1-\nu_2} \frac{\Gamma(1-\nu_1) \Gamma(1-\nu_2) \Gamma(1+\nu_1+\nu_2-D)}{\Gamma\left(1+\nu_2-\frac{D}{2}\right) \Gamma\left(1+\nu_1-\frac{D}{2}\right) \Gamma\left(1+\frac{D}{2}-\nu_1-\nu_2\right)} \\ \times \sum_{m_1, m_2=0}^{\infty} \frac{(1+\nu_1+\nu_2-D, m_1+m_2) \left(\nu_1+\nu_2-\frac{D}{2}, m_1+m_2\right)}{\left(1+\nu_2-\frac{D}{2}, m_2\right) \left(1+\nu_1-\frac{D}{2}, m_1\right)} \frac{(M_1^2/Q_1^2)^{m_1}}{m_1!} \frac{(M_2^2/Q_1^2)^{m_2}}{m_2!}, \quad (2.40)$$

so that we can recognize the general form of Eq. (2.27): the first line of the rhs is  $\mathcal{PRE}$  while the second is  $\mathcal{SUM}$ . By flipping the  $\Gamma$  functions in the prefactor term  $\mathcal{PRE}$ , using Eq. (2.33), we get

$$I_2^{\{m_1, m_2\}} = (-1)^{\frac{D}{2}} \left(Q_1^2\right)^{\frac{D}{2}-\nu_1-\nu_2} \frac{\Gamma\left(\frac{D}{2}-\nu_1\right) \Gamma\left(\frac{D}{2}-\nu_2\right) \Gamma\left(\nu_1+\nu_2-\frac{D}{2}\right)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(D-\nu_1-\nu_2)} \\ \times F_4\left(1+\nu_1+\nu_2-D, \nu_1+\nu_2-\frac{D}{2}, 1+\nu_1-\frac{D}{2}, 1+\nu_2-\frac{D}{2}, \frac{M_1^2}{Q_1^2}, \frac{M_2^2}{Q_1^2}\right), \quad (2.41)$$

where we have used the definition of Appell's  $F_4$  function given in Eq. (A.6).

In the same way, we can obtain the other seven solutions:

$$I_2^{\{p_1, m_1\}} = (-1)^{\frac{D}{2}} \left(Q_1^2\right)^{-\nu_1} \left(-M_2^2\right)^{\frac{D}{2}-\nu_2} \frac{\Gamma\left(\nu_2-\frac{D}{2}\right)}{\Gamma(\nu_2)} \\ \times F_4\left(1+\nu_1-\frac{D}{2}, \nu_1, 1+\nu_1-\frac{D}{2}, 1+\frac{D}{2}-\nu_2, \frac{M_1^2}{Q_1^2}, \frac{M_2^2}{Q_1^2}\right), \\ I_2^{\{p_2, m_2\}} = (-1)^{\frac{D}{2}} \left(Q_1^2\right)^{-\nu_2} \left(-M_1^2\right)^{\frac{D}{2}-\nu_1} \frac{\Gamma\left(\nu_1-\frac{D}{2}\right)}{\Gamma(\nu_1)} \\ \times F_4\left(1+\nu_2-\frac{D}{2}, \nu_2, 1+\frac{D}{2}-\nu_1, 1+\nu_2-\frac{D}{2}, \frac{M_1^2}{Q_1^2}, \frac{M_2^2}{Q_1^2}\right), \\ I_2^{\{p_1, p_2\}} = (-1)^{\frac{D}{2}} \left(Q_1^2\right)^{-\frac{D}{2}} \left(-M_1^2\right)^{\frac{D}{2}-\nu_1} \left(-M_2^2\right)^{\frac{D}{2}-\nu_2} \frac{\Gamma\left(\nu_1-\frac{D}{2}\right) \Gamma\left(\nu_2-\frac{D}{2}\right) \Gamma\left(\frac{D}{2}\right)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(0)} \\ \times F_4\left(1, \frac{D}{2}, 1+\frac{D}{2}-\nu_1, 1+\frac{D}{2}-\nu_2, \frac{M_1^2}{Q_1^2}, \frac{M_2^2}{Q_1^2}\right), \\ I_2^{\{q_1, m_2\}} = (-1)^{\frac{D}{2}} \left(-M_1^2\right)^{\frac{D}{2}-\nu_1-\nu_2} \frac{\Gamma\left(\nu_1+\nu_2-\frac{D}{2}\right) \Gamma\left(\frac{D}{2}-\nu_2\right)}{\Gamma(\nu_1) \Gamma\left(\frac{D}{2}\right)}$$

$$\begin{aligned}
& \times F_4 \left( \nu_1 + \nu_2 - \frac{D}{2}, \nu_2, \frac{D}{2}, 1 + \nu_2 - \frac{D}{2}, \frac{Q_1^2}{M_1^2}, \frac{M_2^2}{M_1^2} \right), \\
I_2^{\{p_1, q_1\}} &= (-1)^{\frac{D}{2}} (-M_1^2)^{-\nu_1} (-M_2^2)^{\frac{D}{2}-\nu_2} \frac{\Gamma(\nu_2 - \frac{D}{2})}{\Gamma(\nu_2)} \\
& \times F_4 \left( \nu_1, \frac{D}{2}, \frac{D}{2}, 1 + \frac{D}{2} - \nu_2, \frac{Q_1^2}{M_1^2}, \frac{M_2^2}{M_1^2} \right), \\
I_2^{\{q_1, m_1\}} &= (-1)^{\frac{D}{2}} (-M_2^2)^{\frac{D}{2}-\nu_1-\nu_2} \frac{\Gamma(\nu_1 + \nu_2 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_1)}{\Gamma(\nu_2) \Gamma(\frac{D}{2})} \\
& \times F_4 \left( \nu_1 + \nu_2 - \frac{D}{2}, \nu_1, 1 + \nu_1 - \frac{D}{2}, \frac{D}{2}, \frac{M_1^2}{M_2^2}, \frac{Q_1^2}{M_2^2} \right), \\
I_2^{\{p_2, q_1\}} &= (-1)^{\frac{D}{2}} (-M_1^2)^{\frac{D}{2}-\nu_1} (-M_2^2)^{-\nu_2} \frac{\Gamma(\nu_1 - \frac{D}{2})}{\Gamma(\nu_1)} \\
& \times F_4 \left( \nu_2, \frac{D}{2}, 1 + \frac{D}{2} - \nu_1, \frac{D}{2}, \frac{M_1^2}{M_2^2}, \frac{Q_1^2}{M_2^2} \right). \tag{2.42}
\end{aligned}$$

Solution  $I_2^{\{p_1, p_2\}}$  deserves a comment. Before any flipping of the  $\Gamma$  functions between numerator and denominator, the prefactor  $\mathcal{PRE}$ , has a  $\Gamma(1)$  in the numerator, due to the fact that the Appell's  $F_4$  function has its first argument equal to 1. Flipping this  $\Gamma(1)$  according to Eq. (2.33) generates a  $\Gamma(0)$  in the denominator, so that this solution is to be considered to be equal to 0.

#### 2.4.1 Identification of the groups of solutions using the convergence regions

The Appell's  $F_4(\alpha, \beta, \gamma, \gamma', x, y)$ , defined in Eq. (A.6), is convergent only if (see Table 1)

$$|\sqrt{x}| + |\sqrt{y}| < 1, \tag{2.43}$$

so that we can form three different groups, according to the kinematic region of convergence of the series

$$\begin{aligned}
I_2^D(\nu_1, \nu_2; Q_1^2, M_1^2, M_2^2) &= I_2^{\{m_1, m_2\}} + I_2^{\{p_2, m_2\}} + I_2^{\{p_1, m_1\}} & \text{if } \sqrt{M_1^2} + \sqrt{M_2^2} < \sqrt{Q_1^2}, \\
I_2^D(\nu_1, \nu_2; Q_1^2, M_1^2, M_2^2) &= I_2^{\{q_1, m_2\}} + I_2^{\{p_1, q_1\}} & \text{if } \sqrt{Q_1^2} + \sqrt{M_2^2} < \sqrt{M_1^2}, \\
I_2^D(\nu_1, \nu_2; Q_1^2, M_1^2, M_2^2) &= I_2^{\{q_1, m_1\}} + I_2^{\{p_2, q_1\}} & \text{if } \sqrt{Q_1^2} + \sqrt{M_1^2} < \sqrt{M_2^2}.
\end{aligned} \tag{2.44}$$

We note that from the convergence properties of the  $F_4$ , if it was not eliminated by the zero in the prefactor,  $I_2^{\{p_1, p_2\}}$  would belong to the first kinematic region,  $\sqrt{M_1^2} + \sqrt{M_2^2} < \sqrt{Q_1^2}$ .



In this way, using NDIM we have simultaneously obtained *all* the different forms of the hypergeometric functions that express the integral  $I_2^D$  for different kinematic regions of  $M_1^2$ ,  $M_2^2$  and  $Q_1^2$ . These results for  $I_2^D$  agree with those obtained with the Mellin-Barnes method [13].

It must be noted that we can go from one kinematic region to the other, just by applying formula (A.51) of the analytic continuation of the  $F_4$  function. As stated at the end of the previous section, the appearance of  $\Gamma(0)$  in the denominator, that occurs during the process of analytic continuation, just kills that term.

## 2.4.2 Identification of the groups of solutions using the system of constraints

We would like to address here a different method to form the groups of solutions. This is based only on considerations of the system of constraints, and more precisely on the sign of the summed indices of the series  $\{p_i, q_i, m_i\}$ , without any knowledge of the region of convergence of the specific series.

For this purpose, the actual value of  $\nu_i$  and of  $D$  in the system (2.26) is irrelevant, because it only modifies the sign of a finite number of the summation variables of the series. For example, solution (2.38), obtained solving the system with respect of the two indices  $\{m_1, m_2\}$ , tells us that the “bulk” sign of  $p_1$  is equal to that of  $m_2$ , because for  $m_2$  sufficiently large, the contribution of  $\nu_2 - \frac{D}{2}$  is no longer important. The same thing happens for  $p_2$  and  $q_1$ , whose “bulk” sign is equal to that of  $m_1$  and  $-m_1 - m_2$ , respectively.

For this reason, instead of considering the full inhomogeneous system, we consider the homogeneous one, obtained by setting  $\nu_i = 0$  and  $D = 0$ . The system (2.26) for the massive bubble then becomes

$$\begin{aligned} q_1 + p_1 + m_1 &= 0, \\ q_1 + p_2 + m_2 &= 0, \\ q_1 + p_1 + p_2 &= 0. \end{aligned} \tag{2.45}$$

We would like to stress the fact that the last equation has always the same form, since this is the constraint expressed by Eq. (2.22).

We can now build a table of signs for  $p_i$ ,  $q_i$  and  $m_i$ . The last equation gives rise to one of the following cases:

$> 0$	$< 0$
$p_1, p_2$	$q_1$
$p_1, q_1$	$p_2$
$p_2, q_1$	$p_1$
$p_1$	$p_2, q_1$
$p_2$	$p_1, q_1$
$q_1$	$p_1, q_2$

Using the last equation of (2.45) to eliminate  $q_1$  from the other two equations of the system (2.45), we have

$$\begin{aligned} m_1 &= p_2, \\ m_2 &= p_1, \end{aligned} \tag{2.46}$$

so that we can complete the previous table in the following way:

$> 0$	$< 0$
$p_1, p_2, m_1, m_2$	$q_1$
$p_1, q_1, m_2$	$p_2, m_1$
$p_2, q_1, m_1$	$p_1, m_2$

where the last three lines have been neglected, being equal and opposite to the first three ones.

We know that the summation indices of the series must be positive integers and we can therefore imagine solving the system (2.26) with respect to any pair of variables that are simultaneously positive. The table provides us with this information and we can directly read from the table which integers are simultaneously positive and use them to form a group of solutions with similar properties by selecting all possible pairs of summation indices from the list of positive indices.

Starting from the first row of the table, and considering the identities (2.46), that embody the fact that we cannot solve the system with respect to the pairs of indices  $\{p_1, m_2\}$  and  $\{p_2, m_1\}$ , because they are linearly dependent, we can form the following subgroups:

$$p_1, p_2, m_1, m_2 \implies \{p_1, p_2\}, \{p_1, m_1\}, \{p_2, m_2\}, \{m_1, m_2\}. \tag{2.47}$$

The same thing can be done with the other two rows of the table:

$$p_1, q_1, m_2 \implies \{p_1, q_1\}, \{q_1, m_2\}, \tag{2.48}$$

$$p_2, q_1, m_1 \implies \{p_2, q_1\}, \{q_1, m_1\}. \tag{2.49}$$

These are exactly the groups obtained by adding solutions according to their region of convergence (see Eq. (2.44)), once we consider the fact that  $I_2^{\{p_1, p_2\}} = 0$ .

This method gives the correct groups only for the cases where the homogeneous system can be solved without any ambiguity. There are examples, and we will meet one in Sec. 3.2.2, where the sign of some indices of the series are undetermined, because they have a dependence on other indices of the type

$$p_1 = p_2 + p_3,$$

where  $p_2$  and  $p_3$  have opposite sign. In this case, we cannot say if  $p_1$  is positive or negative.

Up to now, we do not have a way to deal with these cases directly from the system of constraints, and we leave the task of further investigating this issue to future works.

### 2.4.3 The limiting case: $M_1 \neq 0$ , $M_2 = 0$

We conclude this section, by considering some extreme cases. First we consider the limit of one massless propagator in the self-energy diagram. We can compute this integral in two different ways.

1. We can start with the system (2.26) with  $m_2 = 0$ : we have four variables and three constraints, so that we end up with a single-index series, that turns out to be a Gauss' hypergeometric  ${}_2F_1$  function (see Eq. (A.1)).
2. We can simply take the limit for  $M_2 \rightarrow 0$  of the general expressions (2.41) and (2.42). We can apply this limit only to the solutions that are convergent in the new kinematic regions, and we cannot take the limit for solutions  $I_2^{\{q_1, m_1\}}$  and  $I_2^{\{p_2, q_1\}}$  because they are defined only for  $\sqrt{Q_1^2} + \sqrt{M_1^2} < \sqrt{M_2^2}$ .

The expression for  $F_4(\alpha, \beta, \gamma, \gamma', x, 0)$  is easily obtained from its definition (A.6) with the second summation series collapsing to its first term

$$F_4(\alpha, \beta, \gamma, \gamma', x, 0) = {}_2F_1(\alpha, \beta, \gamma, x). \quad (2.50)$$

Both procedures give the same result.

If  $M_1^2 < Q_1^2$

$$\begin{aligned} I_2^D(\nu_1, \nu_2; Q_1^2, M_1^2, 0) &= I_2^{\{m_1\}} + I_2^{\{p_2\}} \\ &= (-1)^{\frac{D}{2}} (Q_1^2)^{\frac{D}{2} - \nu_1 - \nu_2} \frac{\Gamma(\nu_1 + \nu_2 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_1) \Gamma(\frac{D}{2} - \nu_2)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(D - \nu_1 - \nu_2)} \\ &\times {}_2F_1\left(1 + \nu_1 + \nu_2 - D, \nu_1 + \nu_2 - \frac{D}{2}, 1 + \nu_1 - \frac{D}{2}, \frac{M_1^2}{Q_1^2}\right) \end{aligned}$$

$$\begin{aligned}
& + (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_2} (-M_1^2)^{\frac{D}{2}-\nu_1} \frac{\Gamma(\nu_1 - \frac{D}{2})}{\Gamma(\nu_1)} \\
& \times {}_2F_1\left(\nu_2, 1 + \nu_2 - \frac{D}{2}, 1 + \frac{D}{2} - \nu_1, \frac{M_1^2}{Q_1^2}\right), \tag{2.51}
\end{aligned}$$

if  $Q_1^2 < M_1^2$

$$\begin{aligned}
I_2^D(\nu_1, \nu_2; Q_1^2, M_1^2, 0) &= I_2^{\{q_1\}} \\
&= (-1)^{\frac{D}{2}} (-M_1^2)^{\frac{D}{2}-\nu_1-\nu_2} \frac{\Gamma(\nu_1 + \nu_2 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_2)}{\Gamma(\nu_1) \Gamma(\frac{D}{2})} \\
&\times {}_2F_1\left(\nu_1 + \nu_2 - \frac{D}{2}, \nu_2, \frac{D}{2}, \frac{Q_1^2}{M_1^2}\right). \tag{2.52}
\end{aligned}$$

#### 2.4.4 The limiting case: $Q_1^2 \rightarrow 0$

Similarly, we can take the limit of the general self-energy diagram where the external momentum scale vanishes. Once again, we can either return to the system (2.26) with one fewer variable ( $q_1 = 0$ ) or we just take the  $Q_1^2 \rightarrow 0$  limit of the general result (2.44) in the appropriate kinematic regions:  $\sqrt{Q_1^2} + \sqrt{M_1^2} < \sqrt{M_2^2}$  or  $\sqrt{Q_1^2} + \sqrt{M_2^2} < \sqrt{M_1^2}$ . Both procedures yield the same result:

If  $M_1 > M_2$

$$\begin{aligned}
I_2^D(\nu_1, \nu_2; 0, M_1^2, M_2^2) &= I_2^{\{m_2\}} + I_3^{\{p_1\}} \\
&= (-1)^{\frac{D}{2}} (-M_1^2)^{\frac{D}{2}-\nu_1-\nu_2} \frac{\Gamma(\nu_1 + \nu_2 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_2)}{\Gamma(\nu_1) \Gamma(\frac{D}{2})} \\
&\quad \times {}_2F_1\left(\nu_2, \nu_1 + \nu_2 - \frac{D}{2}, 1 + \nu_2 - \frac{D}{2}, \frac{M_2^2}{M_1^2}\right) \\
&+ (-1)^{\frac{D}{2}} (-M_1^2)^{-\nu_1} (-M_2^2)^{\frac{D}{2}-\nu_2} \frac{\Gamma(\nu_2 - \frac{D}{2})}{\Gamma(\nu_2)} {}_2F_1\left(\nu_1, \frac{D}{2}, 1 + \frac{D}{2} - \nu_2, \frac{M_2^2}{M_1^2}\right), \tag{2.53}
\end{aligned}$$

with the result for  $M_2 > M_1$  obtained by the exchanges  $M_1 \leftrightarrow M_2$  and  $\nu_1 \leftrightarrow \nu_2$ .

Provided that we do not violate the validity of the kinematic regions, we can take the subsequent limits of the energy scales. For example, we can safely take the  $M_2 \rightarrow 0$  limit for the solution where  $M_2 < M_1$ . In this case, only the first term survives and we obtain the familiar result

$$I_2^D(\nu_1, \nu_2; 0, M_1^2, 0) = (-1)^{\frac{D}{2}} (-M_1^2)^{\frac{D}{2}-\nu_1-\nu_2} \frac{\Gamma(\nu_1 + \nu_2 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_2)}{\Gamma(\nu_1) \Gamma(\frac{D}{2})}. \tag{2.54}$$

## 2.5 The special case: all masses equal

For the special case where each propagator has the same mass, Eq. (2.10) becomes

$$\mathcal{M} = \sum_{i=1}^n x_i M^2 = \mathcal{P} M^2, \quad (2.55)$$

and we have an important simplification. As before, we expand the exponentials in Eq. (2.8) and make multinomial expansions of  $\mathcal{Q}$  and  $\mathcal{P}$

$$\begin{aligned} I_n^D(\{\nu_i\}; \{Q_i^2\}, M^2) &= \int \mathcal{D}x \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\mathcal{Q}^n \mathcal{P}^{-n-\frac{D}{2}+m} (-M^2)^m}{n! m!} \\ &= \int \mathcal{D}x \sum_{\substack{p_1, \dots, p_n=0 \\ q_1, \dots, q_q=0 \\ m=0}}^{\infty} \frac{\mathcal{Q}_1^{q_1} \dots \mathcal{Q}_q^{q_q} x_1^{p_1} \dots x_n^{p_n} (-M^2)^m}{q_1! \dots q_q! p_1! \dots p_n! m!} (p_1 + \dots + p_n)!, \end{aligned} \quad (2.56)$$

subject to the constraints

$$\sum_{i=1}^q q_i = n \quad \text{and} \quad \sum_{i=1}^n p_i = -n - \frac{D}{2} + m. \quad (2.57)$$

Equating Eqs. (2.14) and (2.56) and, once again, identifying powers of  $x_i^{-\nu_i}$ , we obtain an expression for the loop integral with negative powers of the propagators in negative dimensions for all masses equal

$$\begin{aligned} I_n^D(\{\nu_i\}; \{Q_i^2\}, M^2) &= \sum_{\substack{p_1, \dots, p_n=0 \\ q_1, \dots, q_q=0 \\ m=0}}^{\infty} (Q_1^2)^{q_1} \dots (Q_q^2)^{q_q} (-M^2)^m \\ &\quad \times \left( \prod_{i=1}^n \frac{\Gamma(1-\nu_i)}{\Gamma(1+p_i)} \right) \left( \prod_{i=1}^q \frac{1}{\Gamma(1+q_i)} \right) \frac{\Gamma(1+\sum_{k=1}^n p_k)}{\Gamma(1+m)}, \end{aligned} \quad (2.58)$$

subject to  $n$  constraints that each of the powers of  $x_i$  match up correctly. However, the constraint that matches up the powers of  $\mathcal{Q}$  and  $\mathcal{P}$ , obtained by summing the two expressions in Eq. (2.57), is now

$$p_1 + \dots + p_n + q_1 + \dots + q_q = -\frac{D}{2} + m, \quad (2.59)$$

rather than Eq. (2.22).

We see that there are  $(n+q+1)$  summation variables and  $(n+1)$  constraints, leaving  $q$  remaining summations. We note that the structure of the solution is precisely as for the unequal-mass case and is treated in the same way by constructing the sum over Pochhammer symbols  $\mathcal{SUM}$  and the  $\Gamma$  function prefactor  $\mathcal{PRE}$ .

**Example:** to give an explicit example, we consider the self-energy correction to the propagator integral with equal masses. There are  $(n + q + 1) = 4$  summation variables with  $(n + 1) = 3$  constraints. In this case, the template solution Eq. (2.58) is given by

$$I_2^D(\nu_1, \nu_2; Q_1^2, M^2, M^2) = \sum_{p_1, p_2, q_1, m=0}^{\infty} (Q_1^2)^{q_1} (-M^2)^m \times \frac{\Gamma(1 - \nu_1) \Gamma(1 - \nu_2) \Gamma(1 + p_1 + p_2)}{\Gamma(1 + p_1) \Gamma(1 + p_2) \Gamma(1 + q_1) \Gamma(1 + m)}, \quad (2.60)$$

while, matching the powers of  $x_i$  gives the system of constraints

$$\begin{aligned} q_1 + p_1 &= -\nu_1, \\ q_1 + p_2 &= -\nu_2, \\ q_1 + p_1 + p_2 &= -\frac{D}{2} + m. \end{aligned} \quad (2.61)$$

There are four summation variables ( $p_1$ ,  $p_2$ ,  $q_1$  and  $m$ ) and three constraints, and we obtain four series solutions, with only one index of summation.

Defining  $\sigma = \nu_1 + \nu_2$  (see Eq. (2.6)), we have:

If  $Q_1^2 > 4M^2$

$$\begin{aligned} I_2^D(\nu_1, \nu_2; Q_1^2, M^2, M^2) &= I_2^{\{m\}} + I_2^{\{p_1\}} + I_2^{\{p_2\}} \\ &= (-1)^{\frac{D}{2}} (Q_1^2)^{\frac{D}{2} - \nu_1 - \nu_2} \frac{\Gamma(\sigma - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_1) \Gamma(\frac{D}{2} - \nu_2)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(D - \nu_1 - \nu_2)} \\ &\quad \times {}_3F_2\left(1 + \frac{\sigma}{2} - \frac{D}{2}, \frac{1}{2} + \frac{\sigma}{2} - \frac{D}{2}, \sigma - \frac{D}{2}, 1 + \nu_1 - \frac{D}{2}, 1 + \nu_2 - \frac{D}{2}, \frac{4M^2}{Q_1^2}\right) \\ &+ (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_1} (-M^2)^{\frac{D}{2} - \nu_2} \frac{\Gamma(\nu_2 - \frac{D}{2})}{\Gamma(\nu_2)} \\ &\quad \times {}_3F_2\left(\nu_1, 1 + \frac{\nu_1}{2} - \frac{\nu_2}{2}, \frac{1}{2} + \frac{\nu_1}{2} - \frac{\nu_2}{2}, 1 + \nu_1 - \nu_2, 1 + \frac{D}{2} - \nu_2, \frac{4M^2}{Q_1^2}\right) \\ &+ (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_2} (-M^2)^{\frac{D}{2} - \nu_1} \frac{\Gamma(\nu_1 - \frac{D}{2})}{\Gamma(\nu_1)} \\ &\quad \times {}_3F_2\left(\nu_2, 1 + \frac{\nu_2}{2} - \frac{\nu_1}{2}, \frac{1}{2} + \frac{\nu_2}{2} - \frac{\nu_1}{2}, 1 + \nu_2 - \nu_1, 1 + \frac{D}{2} - \nu_1, \frac{4M^2}{Q_1^2}\right), \end{aligned} \quad (2.62)$$

if  $Q_1^2 < 4M^2$

$$I_2^D(\nu_1, \nu_2; Q_1^2, M^2, M^2) = I_2^{\{q_1\}}$$

$$= (-1)^{\frac{D}{2}} (-M^2)^{\frac{D}{2}-\nu_1-\nu_2} \frac{\Gamma\left(\sigma - \frac{D}{2}\right)}{\Gamma(\sigma)} {}_3F_2\left(\nu_1, \nu_2, \sigma - \frac{D}{2}, \frac{\sigma}{2}, \frac{1+\sigma}{2}, \frac{Q_1^2}{4M^2}\right). \quad (2.63)$$

In forming the Pochhammer symbols we have made use of the following duplication formula

$$(z, 2n) = 4^n \left(\frac{z}{2}, n\right) \left(\frac{z}{2} + \frac{1}{2}, n\right). \quad (2.64)$$

The procedure described in Sec. 2.4.2 on how the solutions group together to give the correct answer in a particular kinematic region, is straightforward. In fact, the solution of the homogeneous counterpart of the system of constraints (2.61) is

$$p_1 = p_2 = m = -q_1. \quad (2.65)$$

We can then form groups for the indices of summation of the series out of combinations of indices with the same sign, that is:

$$\{p_1\}, \{p_2\}, \{m\} \quad \text{or} \quad \{q_1\}. \quad (2.66)$$

Not surprisingly these were the groups of solutions formed by considering the convergence properties in Eqs. (2.62) and (2.63) and reproduce the known result [13].

### 3 Massive Vertex integrals

We now turn to massive-triangle integrals where each propagator can have a different mass and each external leg can be off-shell. The propagators and momenta are labelled as in Fig. 3. Throughout this section, the number of propagators  $n$  is equal to three and in the most general case, we have

$$\begin{aligned} \mathcal{P} &= x_1 + x_2 + x_3 \\ \mathcal{Q} &= x_2 x_3 Q_1^2 + x_3 x_1 Q_2^2 + x_1 x_2 Q_3^2 \\ \mathcal{M} &= x_1 M_1^2 + x_2 M_2^2 + x_3 M_3^2, \end{aligned} \quad (3.1)$$

where  $Q_i^2 = k_i^2$ .

Based on the discussion of Sec. 2.2, the NDIM method provides a generic solution with  $(n + q + m) = 9$  summation variables and  $(n + 1) = 4$  constraints, with the template solution given by

$$I_3^D \left( \nu_1, \nu_2, \nu_3; Q_1^2, Q_2^2, Q_3^2, M_1^2, M_2^2, M_3^2 \right)$$

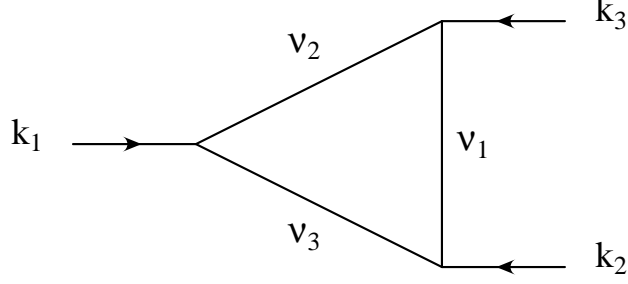


Figure 3: The one-loop vertex diagram

$$\begin{aligned}
&\equiv \sum_{\substack{p_1, \dots, p_3=0 \\ q_1, \dots, q_3=0 \\ m_1, \dots, m_3=0}}^{\infty} (Q_1^2)^{q_1} (Q_2^2)^{q_2} (Q_3^2)^{q_3} (-M_1^2)^{m_1} (-M_2^2)^{m_2} (-M_3^2)^{m_3} \\
&\times \left( \prod_{i=1}^3 \frac{\Gamma(1 - \nu_i)}{\Gamma(1 + m_i) \Gamma(1 + p_i) \Gamma(1 + q_i)} \right) \Gamma(1 + p_1 + p_2 + p_3), \quad (3.2)
\end{aligned}$$

while the system of four constraints is

$$\begin{aligned}
q_2 + q_3 + p_1 + m_1 &= -\nu_1, \\
q_1 + q_3 + p_2 + m_2 &= -\nu_2, \\
q_1 + q_2 + p_3 + m_3 &= -\nu_3, \\
p_1 + p_2 + p_3 + q_1 + q_2 + q_3 &= -\frac{D}{2}. \quad (3.3)
\end{aligned}$$

The number of possible solutions satisfying this system is  $9!/4!/5! = 126$  of which 45 are eliminated by the particular nature of the system leaving 81. As usual, insertion of these solutions into the template yields contributions of the general form (2.27), where  $SUM$  is a product of Pochhammer symbols and ratios of energy scales summed over the five remaining variables. The prefactor  $\mathcal{PRE}$  vanishes in a further 12 instances, leaving 69 solutions which are distributed among the various kinematic regions.

At present, the technology for dealing with five-fold sums (and their integral representations) is not sufficiently developed to handle the completely general case. For the remainder of this section, we therefore concentrate on particular cases of the vertex integral where some of the energy scales vanish, leading to either single or double sums which have been well studied.



### 3.1 Massless propagators: $M_1 = M_2 = M_3 = 0$

We first consider the special case where all of the internal lines are massless, so  $m_1 = m_2 = m_3 = 0$  in Eq. (3.3), leaving six summation variables. Of the  $6!/4!/2! = 15$  possible solutions of this system, three are eliminated by the system, leaving twelve. The three-mass triangle is an extremely symmetric system and there are three allowed phase space regions:

$$\begin{aligned} \text{region I : } & \sqrt{Q_1^2} > \sqrt{Q_2^2} + \sqrt{Q_3^2}, \\ \text{region II : } & \sqrt{Q_2^2} > \sqrt{Q_1^2} + \sqrt{Q_3^2}, \\ \text{region III : } & \sqrt{Q_3^2} > \sqrt{Q_1^2} + \sqrt{Q_2^2}, \end{aligned} \tag{3.4}$$

with each region bounded by the external phase-space constraints

$$\Delta_3(Q_1^2, Q_2^2, Q_3^2) > 0, \tag{3.5}$$

where

$$\Delta_3(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx. \tag{3.6}$$

The twelve solutions populate the three kinematic regions equally, four in each, and are easily identified as belonging to a particular region by studying either the convergence properties of the double sum or by considering the system, as in Sec. 2.4.2. For example, the four solutions belonging to region I ( $q_1$  negative,  $q_2$  and  $q_3$  positive) are those where the summation variables include pairs in the set  $\{p_2, p_3, q_2, q_3\}$ , that is  $\{p_2, p_3\}$ ,  $\{p_2, q_3\}$ ,  $\{q_2, p_3\}$  and  $\{q_2, q_3\}$ , where  $\{p_2, q_2\}$  and  $\{p_3, q_3\}$  have been eliminated by the system.

As usual, each solution is inserted into Eq. (3.2) and treated according to the procedure described in Sec. 2.2: the summation variables are converted into Pochhammer symbols; the  $\Gamma$ -function prefactor is flipped using Eq. (2.33) and the remaining summations converted into generalised hypergeometric functions. In each case, we identify Appell's  $F_4(\alpha, \beta, \gamma, \gamma', x, y)$  function (see Eq. A.6), which, according to the convergence criteria of Table 1, is well defined when  $\sqrt{x} + \sqrt{y} < 1$ , precisely matching on to the physically allowed phase space.

Summing the four solutions we find, in the region  $\sqrt{Q_1^2} > \sqrt{Q_2^2} + \sqrt{Q_3^2}$ ,

$$\begin{aligned} I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, Q_2^2, Q_3^2, 0, 0, 0) &= I_3^{\{q_2, q_3\}} + I_3^{\{p_2, q_3\}} + I_3^{\{p_3, q_2\}} + I_3^{\{p_2, p_3\}} \\ &= (-1)^{\frac{D}{2}} (Q_1^2)^{\frac{D}{2} - \nu_1 - \nu_2 - \nu_3} \frac{\Gamma\left(\frac{D}{2} - \nu_1 - \nu_2\right) \Gamma\left(\frac{D}{2} - \nu_1 - \nu_3\right) \Gamma\left(\sigma - \frac{D}{2}\right)}{\Gamma(\nu_2) \Gamma(\nu_3) \Gamma(D - \sigma)} \\ &\quad \times F_4\left(\nu_1, \sigma - \frac{D}{2}, 1 + \nu_1 + \nu_3 - \frac{D}{2}, 1 + \nu_1 + \nu_2 - \frac{D}{2}, \frac{Q_2^2}{Q_1^2}, \frac{Q_3^2}{Q_1^2}\right) \\ &+ (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_2} (Q_2^2)^{\frac{D}{2} - \nu_1 - \nu_3} \frac{\Gamma\left(\frac{D}{2} - \nu_1 - \nu_2\right) \Gamma\left(\nu_1 + \nu_3 - \frac{D}{2}\right) \Gamma\left(\frac{D}{2} - \nu_3\right)}{\Gamma(\nu_1) \Gamma(\nu_3) \Gamma(D - \sigma)} \end{aligned}$$

$$\begin{aligned}
& \times F_4 \left( \nu_2, \frac{D}{2} - \nu_3, 1 + \frac{D}{2} - \nu_1 - \nu_3, 1 + \nu_1 + \nu_2 - \frac{D}{2}, \frac{Q_2^2}{Q_1^2}, \frac{Q_3^2}{Q_1^2} \right) \\
& + (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_3} (Q_3^2)^{\frac{D}{2} - \nu_1 - \nu_2} \frac{\Gamma \left( \frac{D}{2} - \nu_1 - \nu_3 \right) \Gamma \left( \nu_1 + \nu_2 - \frac{D}{2} \right) \Gamma \left( \frac{D}{2} - \nu_2 \right)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(D - \sigma)} \\
& \times F_4 \left( \nu_3, \frac{D}{2} - \nu_2, 1 + \nu_1 + \nu_3 - \frac{D}{2}, 1 + \frac{D}{2} - \nu_1 - \nu_2, \frac{Q_2^2}{Q_1^2}, \frac{Q_3^2}{Q_1^2} \right) \\
& + (-1)^{\frac{D}{2}} (Q_1^2)^{\nu_1 - \frac{D}{2}} (Q_2^2)^{\frac{D}{2} - \nu_1 - \nu_3} (Q_3^2)^{\frac{D}{2} - \nu_1 - \nu_2} \\
& \times \frac{\Gamma \left( \nu_1 + \nu_2 - \frac{D}{2} \right) \Gamma \left( \nu_1 + \nu_3 - \frac{D}{2} \right) \Gamma \left( \frac{D}{2} - \nu_1 \right)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(\nu_3)} \\
& \times F_4 \left( D - \sigma, \frac{D}{2} - \nu_1, 1 + \frac{D}{2} - \nu_1 - \nu_3, 1 + \frac{D}{2} - \nu_1 - \nu_2, \frac{Q_2^2}{Q_1^2}, \frac{Q_3^2}{Q_1^2} \right), \quad (3.7)
\end{aligned}$$

which agrees with that obtained by Boos and Davydychev [13] using the Mellin-Barnes integral representation. Similar results are obtained for the other two kinematic regions, either by directly summing the solutions valid in that region (pairs from  $\{p_1, p_3, q_1, q_3\}$  or  $\{p_1, p_2, q_1, q_2\}$ , with  $\{p_1, q_1\}$ ,  $\{p_2, q_2\}$  and  $\{p_3, q_3\}$  excluded by the system) or by analytic continuation of the Appell's  $F_4$  function using formula (A.51).

Note that if one of the  $\nu_i$  vanishes (equivalent to propagator  $i$  shrinking to a point), only a single term remains. For example, if  $\nu_1 = 0$ , only the first term of Eq. (3.7) survives (the others being killed by  $1/\Gamma(0)$ ), and the Appell's function collapses to

$$F_4(0, \beta, \gamma, \gamma', x, y) = 1,$$

as can be seen from the definition (A.6), yielding

$$I_3^D(0, \nu_2, \nu_3; Q_1^2, Q_2^2, Q_3^2, 0, 0, 0) = I_2^D(\nu_2, \nu_3; Q_1^2, 0, 0), \quad (3.8)$$

as it should.

We can obtain some other interesting limits if we set to zero one or two external invariants.

1. **One light-like external momentum:** if the  $i$ th external leg is light-like ( $Q_i^2 = 0$ ), we can return to the general case and solve the system (3.3) with  $q_i = 0$ , or we can take the appropriate limit of the general solution. These limits can be safely made provided that we start from a valid kinematic region. In the region of validity of Eq. (3.7), that is  $\sqrt{Q_1^2} > \sqrt{Q_2^2} + \sqrt{Q_3^2}$ , we can surely take the limits for  $Q_2^2 \rightarrow 0$  or  $Q_3^2 \rightarrow 0$ . In this last case, for example, the last two terms in Eq. (3.7) vanish, while the first two terms collapse to Gaussian hypergeometric functions, according to Eq. (A.20), yielding, in

the region  $Q_1^2 > Q_2^2$ ,

$$\begin{aligned}
I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, Q_2^2, 0, 0, 0, 0) &= I_3^{\{q_2\}} + I_3^{\{p_2\}} \\
&= (-1)^{\frac{D}{2}} (Q_1^2)^{\frac{D}{2}-\nu_1-\nu_2-\nu_3} \frac{\Gamma\left(\frac{D}{2}-\nu_1-\nu_2\right) \Gamma\left(\frac{D}{2}-\nu_1-\nu_3\right) \Gamma\left(\nu_1+\nu_2+\nu_3-\frac{D}{2}\right)}{\Gamma(\nu_2) \Gamma(\nu_3) \Gamma(D-\nu_1-\nu_2-\nu_3)} \\
&\quad \times {}_2F_1\left(\nu_1, \sigma - \frac{D}{2}, 1 + \nu_1 + \nu_3 - \frac{D}{2}, \frac{Q_2^2}{Q_1^2}\right) \\
&+ (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_2} (Q_2^2)^{\frac{D}{2}-\nu_1-\nu_3} \frac{\Gamma\left(\frac{D}{2}-\nu_1-\nu_2\right) \Gamma\left(\frac{D}{2}-\nu_3\right) \Gamma\left(\nu_1+\nu_3-\frac{D}{2}\right)}{\Gamma(\nu_1) \Gamma(\nu_3) \Gamma(D-\nu_1-\nu_2-\nu_3)} \\
&\quad \times {}_2F_1\left(\nu_2, \frac{D}{2}-\nu_3, 1 + \frac{D}{2}-\nu_1-\nu_3, \frac{Q_2^2}{Q_1^2}\right). \tag{3.9}
\end{aligned}$$

Analogous results valid in the region  $Q_2^2 > Q_1^2$  can be obtained either by starting from the expression for  $I_3^D$  in region II, that is  $\sqrt{Q_2^2} > \sqrt{Q_1^2} + \sqrt{Q_3^2}$ , or via analytic continuation of Eq. (3.9), according to Eq. (A.49).

2. **Two light-like external momenta:** in a similar way, we can obtain the result for two light-like external momenta,  $Q_3^2 = Q_2^2 = 0$  for example, by simultaneously taking both  $Q_2^2$  and  $Q_3^2 \rightarrow 0$  in Eq. (3.7). Only the first term in Eq. (3.7) survives, and the Appell's function collapses to

$$F_4(\alpha, \beta, \gamma, \gamma', 0, 0) = 1, \tag{3.10}$$

yielding

$$\begin{aligned}
I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, 0, 0, 0) &= (-1)^{\frac{D}{2}} (Q_1^2)^{\frac{D}{2}-\sigma} \\
&\times \frac{\Gamma\left(\frac{D}{2}-\nu_1-\nu_2\right) \Gamma\left(\frac{D}{2}-\nu_1-\nu_3\right) \Gamma\left(\sigma-\frac{D}{2}\right)}{\Gamma(\nu_2) \Gamma(\nu_3) \Gamma(D-\sigma)}, \tag{3.11}
\end{aligned}$$

which again agrees with the known result straightforwardly obtained using Feynman parameters. Alternatively, we could have returned to the general system (3.3), where, with only the  $Q_1^2$  scale ( $q_2 = q_3 = m_1 = m_2 = m_3 = 0$ ), we would have had  $(n+q+m) = 4$  summation variables and  $(n+1) = 4$  constraints.

### 3.2 Two massive propagators and one off-shell leg

We now turn to triangle integrals with two internal mass scales and one external scale. These have  $(n+q+m) = 6$  summation variables and  $(n+1) = 4$  constraints and are therefore described by double sums.

### 3.2.1 $M_1 = 0, Q_2^2 = Q_3^2 = 0$

In this case, the system of constraints is obtained by setting  $m_1 = q_2 = q_3 = 0$  in Eq. (3.3). The first constraint is simply  $p_1 = -\nu_1$  and there are only 8 solutions for the system. As usual, each solution is inserted into Eq. (3.2) and treated accordingly to the procedure of Sec 2.2.

The solutions can be grouped either by studying the physical thresholds of the integral (or the convergence properties of the series) or by considering the system, as in Sec. 2.4.2.

The threshold for the production of two massive propagators on-shell,  $\sqrt{Q_1^2} = M_2 + M_3$ , becomes evident when we inspect the hypergeometric functions: four solutions are convergent above threshold,  $\sqrt{Q_1^2} > \sqrt{M_2^2} + \sqrt{M_3^2}$ , while the other solutions equally populate the regions  $\sqrt{M_3^2} > \sqrt{M_2^2} + \sqrt{Q_1^2}$  and  $\sqrt{M_2^2} > \sqrt{M_3^2} + \sqrt{Q_1^2}$ .

Consideration of the system reveals that the first group of solutions are pairs from the set  $\{m_2, m_3, p_2, p_3\}$  and the other groups formed are from the sets  $\{q_1, p_3, m_2\}$  and  $\{q_1, p_2, m_3\}$  respectively. The apparent overlap between the groups, solutions formed from the pairs  $\{m_2, p_3\}$  and  $\{m_3, p_2\}$  are excluded by the system. In each case, we identify Appell's  $F_4$  function, whose convergence properties match onto the anticipated regions.

We find:

$$\begin{aligned}
& \text{If } \sqrt{Q_1^2} > \sqrt{M_2^2} + \sqrt{M_3^2} \\
& I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, 0, M_2^2, M_3^2) = I_3^{\{m_2, m_3\}} + I_3^{\{m_2, p_2\}} + I_3^{\{m_3, p_3\}} + I_3^{\{p_2, p_3\}} \\
& = (-1)^{\frac{D}{2}} (Q_1^2)^{\frac{D}{2} - \nu_1 - \nu_2 - \nu_3} \frac{\Gamma(\nu_1 + \nu_2 + \nu_3 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_1 - \nu_2) \Gamma(\frac{D}{2} - \nu_1 - \nu_3)}{\Gamma(\nu_2) \Gamma(\nu_3) \Gamma(D - \nu_1 - \nu_2 - \nu_3)} \\
& \quad \times F_4\left(1 + \sigma - D, \sigma - \frac{D}{2}, 1 + \nu_1 + \nu_2 - \frac{D}{2}, 1 + \nu_1 + \nu_3 - \frac{D}{2}, \frac{M_2^2}{Q_1^2}, \frac{M_3^2}{Q_1^2}\right) \\
& + (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_2} (-M_3^2)^{\frac{D}{2} - \nu_1 - \nu_3} \frac{\Gamma(\nu_1 + \nu_3 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_1 - \nu_2)}{\Gamma(\nu_3) \Gamma(\frac{D}{2} - \nu_2)} \\
& \quad \times F_4\left(1 + \nu_2 - \frac{D}{2}, \nu_2, 1 + \nu_1 + \nu_2 - \frac{D}{2}, 1 + \frac{D}{2} - \nu_1 - \nu_3, \frac{M_2^2}{Q_1^2}, \frac{M_3^2}{Q_1^2}\right) \\
& + (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_3} (-M_2^2)^{\frac{D}{2} - \nu_1 - \nu_2} \frac{\Gamma(\nu_1 + \nu_2 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_1 - \nu_3)}{\Gamma(\nu_2) \Gamma(\frac{D}{2} - \nu_3)} \\
& \quad \times F_4\left(1 + \nu_3 - \frac{D}{2}, \nu_3, 1 + \frac{D}{2} - \nu_1 - \nu_2, 1 + \nu_1 + \nu_3 - \frac{D}{2}, \frac{M_2^2}{Q_1^2}, \frac{M_3^2}{Q_1^2}\right) \\
& + (-1)^{\frac{D}{2}} (Q_1^2)^{\nu_1 - \frac{D}{2}} (-M_2^2)^{\frac{D}{2} - \nu_1 - \nu_2} (-M_3^2)^{\frac{D}{2} - \nu_1 - \nu_3}
\end{aligned}$$

$$\begin{aligned}
& \times \frac{\Gamma\left(\nu_1 + \nu_2 - \frac{D}{2}\right) \Gamma\left(\nu_1 + \nu_3 - \frac{D}{2}\right) \Gamma\left(\frac{D}{2} - \nu_1\right)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(\nu_3)} \\
& \times F_4\left(1 - \nu_1, \frac{D}{2} - \nu_1, 1 + \frac{D}{2} - \nu_1 - \nu_2, 1 + \frac{D}{2} - \nu_1 - \nu_3, \frac{M_2^2}{Q_1^2}, \frac{M_3^2}{Q_1^2}\right), \quad (3.12)
\end{aligned}$$

if  $\sqrt{M_3^2} > \sqrt{Q_1^2} + \sqrt{M_2^2}$

$$\begin{aligned}
I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, 0, M_2^2, M_3^2) &= I_3^{\{m_2, q_1\}} + I_3^{\{p_3, q_1\}} \\
&= (-1)^{\frac{D}{2}} (-M_3^2)^{\frac{D}{2} - \nu_1 - \nu_2 - \nu_3} \frac{\Gamma\left(\nu_1 + \nu_2 + \nu_3 - \frac{D}{2}\right) \Gamma\left(\frac{D}{2} - \nu_1 - \nu_2\right)}{\Gamma(\nu_3) \Gamma\left(\frac{D}{2}\right)} \\
&\times F_4\left(\nu_1 + \nu_2 + \nu_3 - \frac{D}{2}, \nu_2, 1 + \nu_1 + \nu_2 - \frac{D}{2}, \frac{D}{2}, \frac{M_2^2}{M_3^2}, \frac{Q_1^2}{M_3^2}\right) \\
&+ (-1)^{\frac{D}{2}} (-M_2^2)^{\frac{D}{2} - \nu_1 - \nu_2} (-M_3^2)^{-\nu_3} \frac{\Gamma\left(\nu_1 + \nu_2 - \frac{D}{2}\right) \Gamma\left(\frac{D}{2} - \nu_1\right)}{\Gamma(\nu_2) \Gamma\left(\frac{D}{2}\right)} \\
&\times F_4\left(\frac{D}{2} - \nu_1, \nu_3, 1 + \frac{D}{2} - \nu_1 - \nu_2, \frac{D}{2}, \frac{M_2^2}{M_3^2}, \frac{Q_1^2}{M_3^2}\right), \quad (3.13)
\end{aligned}$$

while the result for  $\sqrt{M_2^2} > \sqrt{M_3^2} + \sqrt{Q_1^2}$  is obtained by the exchanges  $M_2 \leftrightarrow M_3$ ,  $\nu_2 \leftrightarrow \nu_3$  in Eq. (3.13).

We can check that these expressions are valid in certain limits.

## Checks

### - The $\nu_1 \rightarrow 0$ limit: $I_3^D(0, \nu_2, \nu_3; Q_1^2, 0, 0, 0, M_2^2, M_3^2)$

Pinching out the first propagator, the first three terms in (3.12) and both terms in (3.13) survive, yielding the general bubble integral of Eq. (2.44), in the respective kinematic regions,

$$I_3^D(0, \nu_2, \nu_3; Q_1^2, 0, 0, 0, M_2^2, M_3^2) = I_2^D(\nu_2, \nu_3; Q_1^2, M_2^2, M_3^2). \quad (3.14)$$

### - The $\nu_2 \rightarrow 0$ limit: $I_3^D(\nu_1, 0, \nu_3; Q_1^2, 0, 0, 0, M_2^2, M_3^2)$

This limit should produce a self-energy integral with no external momentum and a single internal mass  $M_3$ . This is indeed the case: only the second term in (3.12) and the first term of (3.13) survive, each yielding the same result of Eq. (2.54).

### - The $M_2 \rightarrow 0$ limit: $I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, 0, 0, M_3^2)$

Here the real production threshold occurs at  $Q_1^2 = M_3^2$  and in this limit, Eq. (3.12)

provides the  $Q_1^2 > M_3^2$  result, while Eq. (3.13) gives the expression for  $M_3^2 > Q_1^2$ . The Appell functions again collapse to form Gaussian hypergeometric functions (see Eq. (A.21)), and we find:

$$\begin{aligned}
& \text{If } \sqrt{Q_1^2} > \sqrt{M_3^2} \\
& I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, 0, 0, M_3^2) = I_3^{\{m_3\}} + I_3^{\{p_3\}} \\
& = (-1)^{\frac{D}{2}} (Q_1^2)^{\frac{D}{2}-\nu_1-\nu_2-\nu_3} \frac{\Gamma\left(\sigma - \frac{D}{2}\right) \Gamma\left(\frac{D}{2} - \nu_1 - \nu_2\right) \Gamma\left(\frac{D}{2} - \nu_1 - \nu_3\right)}{\Gamma(\nu_2) \Gamma(\nu_3) \Gamma(D - \sigma)} \\
& \quad \times {}_2F_1\left(1 + \sigma - D, \sigma - \frac{D}{2}, 1 + \nu_1 + \nu_3 - \frac{D}{2}, \frac{M_3^2}{Q_1^2}\right) \\
& + (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_2} (-M_3^2)^{\frac{D}{2}-\nu_1-\nu_3} \frac{\Gamma\left(\nu_1 + \nu_3 - \frac{D}{2}\right) \Gamma\left(\frac{D}{2} - \nu_1 - \nu_2\right)}{\Gamma(\nu_3) \Gamma\left(\frac{D}{2} - \nu_2\right)} \\
& \quad \times {}_2F_1\left(\nu_2, 1 + \nu_2 - \frac{D}{2}, 1 + \frac{D}{2} - \nu_1 - \nu_3, \frac{M_3^2}{Q_1^2}\right), \tag{3.15}
\end{aligned}$$

$$\begin{aligned}
& \text{if } \sqrt{M_3^2} > \sqrt{Q_1^2} \\
& I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, 0, 0, M_3^2) = I_3^{\{q_1\}} \\
& = (-1)^{\frac{D}{2}} (-M_3^2)^{\frac{D}{2}-\nu_1-\nu_2-\nu_3} \frac{\Gamma\left(\sigma - \frac{D}{2}\right) \Gamma\left(\frac{D}{2} - \nu_1 - \nu_2\right)}{\Gamma(\nu_3) \Gamma\left(\frac{D}{2}\right)} {}_2F_1\left(\nu_2, \sigma - \frac{D}{2}, \frac{D}{2}, \frac{Q_1^2}{M_3^2}\right). \tag{3.16}
\end{aligned}$$

This latter result agrees with that obtained by taking the limit  $Q_2^2 \rightarrow 0$ ,  $Q_3^2 \rightarrow 0$  in the general result given by Boos and Davydychev [13] for a triangle loop integral with a single massive propagator.

### 3.2.2 $M_3 = 0$ , $Q_2^2 = Q_3^2 = 0$

We now consider the triangle graph where  $M_3 = 0$  and  $Q_2^2 = Q_3^2 = 0$ . Although this graph is not usually present in Standard Model processes, the analysis of this graph turns out to be rather more subtle than the preceding triangle integrals and we will therefore describe it in more detail. Inspection of the singularities present in the loop integral via the Landau equations reveals that threshold singularities occur at  $M_2^2 = Q_1^2 + M_1^2$ . We expect that this equality will provide the appropriate boundaries of regions of convergence when considering the convergence properties of the generalised hypergeometric functions. Furthermore, since the convergence properties of these functions only depend on the absolute value of ratios of

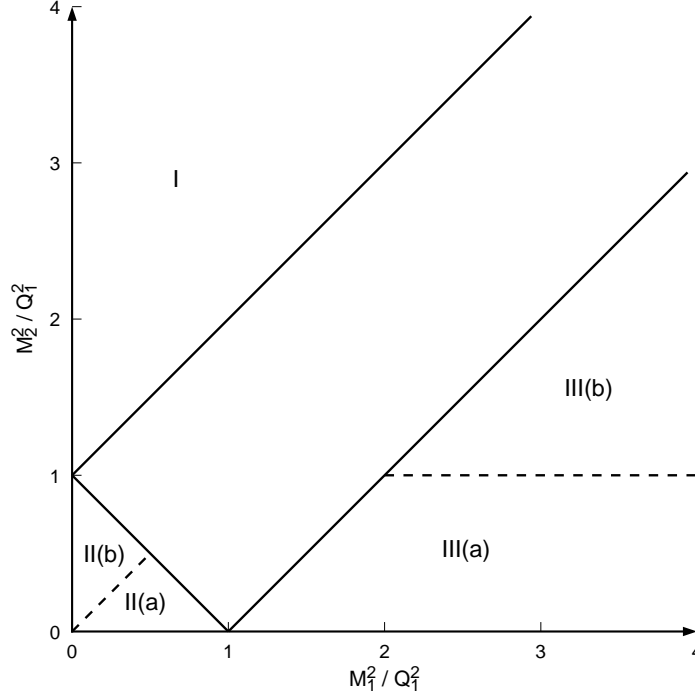


Figure 4: The kinematic regions for the one-loop triangle with  $Q_2^2 = Q_3^2 = M_3^2 = 0$ . The solid line shows the threshold in the Landau surface at  $M_2^2 = Q_1^2 + M_1^2$ , together with the reflections  $M_1^2 + M_2^2 = Q_1^2$  and  $M_1^2 = M_2^2 + Q_1^2$ . The reflections are relevant for the convergence properties of the hypergeometric functions which only involve the absolute values of ratios of the scales. The dashed lines show the boundaries  $M_1^2 = M_2^2$  and  $M_2^2 = Q_1^2$ .

scales, we expect that the reflections,  $M_1^2 + M_2^2 = Q_1^2$  and  $M_1^2 = M_2^2 + Q_1^2$ , will also form boundaries in the large  $Q_1^2$  and  $M_1^2$  regions, respectively. We also expect that, in certain limits, the solutions match onto the kinematic regions relevant for simpler integrals. For example, in the limit  $M_1 \rightarrow 0$ , the discussion of the previous section informs us that the solutions divide according to whether or not  $Q_1^2 > M_2^2$ . Similarly as  $Q_1^2 \rightarrow 0$ , there should be a threshold at  $M_1 = M_2$ .

In anticipation, we therefore divide the kinematic regions up as follows:

$$\begin{aligned}
\text{region I :} & \quad M_2^2 > Q_1^2 + M_1^2, \\
\text{region II(a) :} & \quad Q_1^2 > M_1^2 + M_2^2 \text{ and } M_1^2 > M_2^2, \\
\text{region II(b) :} & \quad Q_1^2 > M_1^2 + M_2^2 \text{ and } M_2^2 > M_1^2, \\
\text{region III(a) :} & \quad M_1^2 > Q_1^2 + M_2^2 \text{ and } Q_1^2 > M_2^2, \\
\text{region III(b) :} & \quad M_1^2 > Q_1^2 + M_2^2 \text{ and } M_2^2 > Q_1^2,
\end{aligned} \tag{3.17}$$

as shown in Fig. 4.

With this set of scales, the system is given by Eq. (3.3) with  $m_3 = q_2 = q_3 = 0$ . As usual, we construct the solutions by solving the system, inserting the solutions into Eq. (3.2) and following the procedure outlined in Sec 2.2. Labelling each solution by the summation variables and using the definitions of the hypergeometric functions of Sec. A.1, we find:

$$\begin{aligned}
I_3^{\{m_1, q_1\}} &= (-1)^{\frac{D}{2}} (-M_2^2)^{\frac{D}{2}-\nu_1-\nu_2-\nu_3} \frac{\Gamma(\nu_1 + \nu_2 + \nu_3 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_1 - \nu_3)}{\Gamma(\nu_2) \Gamma(\frac{D}{2})} \\
&\quad \times F_2 \left( \nu_1 + \nu_2 + \nu_3 - \frac{D}{2}, \nu_1, \nu_3, 1 + \nu_1 + \nu_3 - \frac{D}{2}, \frac{D}{2}, \frac{M_1^2}{M_2^2}, \frac{Q_1^2}{M_2^2} \right), \\
I_3^{\{p_2, q_1\}} &= (-1)^{\frac{D}{2}} (-M_1^2)^{\frac{D}{2}-\nu_1-\nu_3} (-M_2^2)^{-\nu_2} \frac{\Gamma(\nu_1 + \nu_3 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_3)}{\Gamma(\nu_1) \Gamma(\frac{D}{2})} \\
&\quad \times F_2 \left( \nu_2, \frac{D}{2} - \nu_3, \nu_3, 1 + \frac{D}{2} - \nu_1 - \nu_3, \frac{D}{2}, \frac{M_1^2}{M_2^2}, \frac{Q_1^2}{M_2^2} \right), \\
I_3^{\{m_1, m_2\}} &= (-1)^{\frac{D}{2}} (Q_1^2)^{\frac{D}{2}-\nu_1-\nu_2-\nu_3} \frac{\Gamma(\nu_1 + \nu_2 + \nu_3 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_1 - \nu_2) \Gamma(\frac{D}{2} - \nu_1 - \nu_3)}{\Gamma(\nu_2) \Gamma(\nu_3) \Gamma(D - \nu_1 - \nu_2 - \nu_3)} \\
&\quad \times S_1 \left( 1 + \sigma - D, \sigma - \frac{D}{2}, \nu_1, 1 + \nu_1 + \nu_2 - \frac{D}{2}, 1 + \nu_1 + \nu_3 - \frac{D}{2}, -\frac{M_1^2}{Q_1^2}, \frac{M_2^2}{Q_1^2} \right), \\
I_3^{\{m_2, p_2\}} &= (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_2} (-M_1^2)^{\frac{D}{2}-\nu_1-\nu_3} \frac{\Gamma(\nu_1 + \nu_3 - \frac{D}{2}) \Gamma(\nu_3 - \nu_2) \Gamma(\frac{D}{2} - \nu_3)}{\Gamma(\nu_1) \Gamma(\nu_3) \Gamma(\frac{D}{2} - \nu_2)} \\
&\quad \times S_1 \left( 1 + \nu_2 - \frac{D}{2}, \nu_2, \frac{D}{2} - \nu_3, 1 + \nu_2 - \nu_3, 1 + \frac{D}{2} - \nu_1 - \nu_3, -\frac{M_1^2}{Q_1^2}, \frac{M_2^2}{Q_1^2} \right), \\
I_3^{\{m_2, p_3\}} &= (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_3} (-M_1^2)^{\frac{D}{2}-\nu_1-\nu_2} \frac{\Gamma(\nu_1 + \nu_2 - \frac{D}{2}) \Gamma(\nu_2 - \nu_3) \Gamma(\frac{D}{2} - \nu_2)}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(\frac{D}{2} - \nu_3)} \\
&\quad \times S_2 \left( \nu_2 - \nu_3, \nu_1 + \nu_2 - \frac{D}{2}, 1 + \nu_3 - \frac{D}{2}, \nu_3, 1 + \nu_2 - \frac{D}{2}, \frac{M_2^2}{M_1^2}, \frac{M_1^2}{Q_1^2} \right), \\
I_3^{\{p_1, p_3\}} &= (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_3} (-M_1^2)^{-\nu_1} (-M_2^2)^{\frac{D}{2}-\nu_2} \frac{\Gamma(\nu_2 - \frac{D}{2})}{\Gamma(\nu_2)} \\
&\quad \times F_3 \left( \nu_1, 1 + \nu_3 - \frac{D}{2}, \frac{D}{2} - \nu_3, \nu_3, 1 + \frac{D}{2} - \nu_2, \frac{M_2^2}{M_1^2}, \frac{M_2^2}{Q_1^2} \right), \\
I_3^{\{m_1, p_3\}} &= (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_3} (-M_2^2)^{\frac{D}{2}-\nu_1-\nu_2} \frac{\Gamma(\nu_1 + \nu_2 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_1 - \nu_3)}{\Gamma(\nu_2) \Gamma(\frac{D}{2} - \nu_3)} \\
&\quad \times H_2 \left( \nu_1 + \nu_2 - \frac{D}{2}, \nu_1, 1 + \nu_3 - \frac{D}{2}, \nu_3, 1 + \nu_1 + \nu_3 - \frac{D}{2}, \frac{M_1^2}{M_2^2}, -\frac{M_2^2}{Q_1^2} \right),
\end{aligned}$$



$$\begin{aligned}
I_3^{\{p_2, p_3\}} &= (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_3} (-M_1^2)^{\frac{D}{2}-\nu_1-\nu_3} (-M_2^2)^{\nu_3-\nu_2} \frac{\Gamma(\nu_1 + \nu_3 - \frac{D}{2}) \Gamma(\nu_2 - \nu_3)}{\Gamma(\nu_1) \Gamma(\nu_2)} \\
&\quad \times H_2\left(\nu_2 - \nu_3, \frac{D}{2} - \nu_3, 1 + \nu_3 - \frac{D}{2}, \nu_3, 1 + \frac{D}{2} - \nu_1 - \nu_3, \frac{M_1^2}{M_2^2}, -\frac{M_2^2}{Q_1^2}\right), \\
I_3^{\{m_2, p_1\}} &= (-1)^{\frac{D}{2}} (Q_1^2)^{\frac{D}{2}-\nu_2-\nu_3} (-M_1^2)^{-\nu_1} \frac{\Gamma(\nu_2 + \nu_3 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_2) \Gamma(\frac{D}{2} - \nu_3)}{\Gamma(\nu_2) \Gamma(\nu_3) \Gamma(D - \nu_2 - \nu_3)} \\
&\quad \times S_2\left(\nu_2 + \nu_3 - \frac{D}{2}, 1 + \nu_2 + \nu_3 - D, \nu_1, \frac{D}{2} - \nu_3, 1 + \nu_2 - \frac{D}{2}, \frac{M_2^2}{Q_1^2}, \frac{Q_1^2}{M_1^2}\right), \\
I_3^{\{m_2, q_1\}} &= (-1)^{\frac{D}{2}} (-M_1^2)^{\frac{D}{2}-\nu_1-\nu_2-\nu_3} \frac{\Gamma(\nu_1 + \nu_2 + \nu_3 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_2 - \nu_3)}{\Gamma(\nu_1) \Gamma(\frac{D}{2})} \\
&\quad \times S_1\left(\nu_2, \nu_1 + \nu_2 + \nu_3 - \frac{D}{2}, \nu_3, 1 + \nu_2 + \nu_3 - \frac{D}{2}, \frac{D}{2}, -\frac{Q_1^2}{M_1^2}, \frac{M_2^2}{M_1^2}\right), \\
I_3^{\{p_1, q_1\}} &= (-1)^{\frac{D}{2}} (-M_1^2)^{-\nu_1} (-M_2^2)^{\frac{D}{2}-\nu_2-\nu_3} \frac{\Gamma(\nu_2 + \nu_3 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_3)}{\Gamma(\nu_2) \Gamma(\frac{D}{2})} \\
&\quad \times H_2\left(\nu_2 + \nu_3 - \frac{D}{2}, \nu_3, \nu_1, \frac{D}{2} - \nu_3, \frac{D}{2}, \frac{Q_1^2}{M_2^2}, -\frac{M_2^2}{M_1^2}\right). \tag{3.18}
\end{aligned}$$

We now need to study the convergence properties of these solutions. For example, by inspecting Table 1 in Appendix A.1, we see that the function  $S_1(\dots, x, y)$  is convergent when  $|x| + |y| < 1$ . This implies that solution  $I_3^{\{m_1, m_2\}}$  is convergent when

$$\left| \frac{-M_1^2}{Q_1^2} \right| + \left| \frac{M_2^2}{Q_1^2} \right| < 1, \tag{3.19}$$

or, in other words,

$$M_1^2 + M_2^2 < Q_1^2, \tag{3.20}$$

independently of whether  $M_1$  is larger than  $M_2$  or not. This series therefore converges in both regions II(a) and II(b).

On the other hand,  $I_3^{\{m_1, p_3\}}$  converges when

$$-\left| \frac{M_1^2}{M_2^2} \right| + \left| \frac{Q_1^2}{M_2^2} \right| > 1 \quad \text{and} \quad \left| \frac{M_1^2}{M_2^2} \right| < 1 \quad \text{and} \quad \left| \frac{M_2^2}{Q_1^2} \right| < 1, \tag{3.21}$$

or, alternatively,

$$M_1^2 + M_2^2 < Q_1^2 \quad \text{and} \quad M_2^2 > M_1^2, \tag{3.22}$$

which corresponds to region II(b) only.

Applying the convergence criteria to each of the eleven solutions, we find that they are distributed as follows:

in region I

$$I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, M_1^2, M_2^2, 0) = I_3^{\{m_1, q_1\}} + I_3^{\{p_2, q_1\}}, \quad (3.23)$$

in region II(a)

$$I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, M_1^2, M_2^2, 0) = I_3^{\{m_1, m_2\}} + I_3^{\{m_2, p_2\}} + I_3^{\{m_2, p_3\}} + I_3^{\{p_1, p_3\}}, \quad (3.24)$$

in region II(b)

$$I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, M_1^2, M_2^2, 0) = I_3^{\{m_1, m_2\}} + I_3^{\{m_2, p_2\}} + I_3^{\{m_1, p_3\}} + I_3^{\{p_2, p_3\}}, \quad (3.25)$$

in region III(a)

$$I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, M_1^2, M_2^2, 0) = I_3^{\{m_2, p_1\}} + I_3^{\{m_2, q_1\}} + I_3^{\{p_1, p_3\}}, \quad (3.26)$$

in region III(b)

$$I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, M_1^2, M_2^2, 0) = I_3^{\{m_2, q_1\}} + I_3^{\{p_1, q_1\}}. \quad (3.27)$$

We see that in region II(a), two of the solutions ( $I_3^{\{m_2, p_2\}}$  and  $I_3^{\{m_2, p_3\}}$ ) contain dangerous  $\Gamma$  functions when  $\nu_2 = \nu_3$ . These divergences usually indicate the region of a logarithmic analytic continuation and can be regulated by letting  $\nu_2 = \nu_3 + \delta$ , canceling the divergence, and then setting  $\delta \rightarrow 0$ . Similarly, the two divergent contributions in region II(b) ( $I_3^{\{m_2, p_2\}}$  and  $I_3^{\{p_2, p_3\}}$ ) also cancel in this limit.

We have performed several checks of the correctness of this assignment into groups.

## Checks

### - Analytic continuation

Applying the analytic continuation formulae given in Appendix A, we can see that the solutions are connected to each other. For example, applying Eq. (A.54) to the  $F_2$  functions in region I produces the  $H_2$  and  $S_1$  functions of region II(b). Similarly, Eqs. (A.56) and (A.53) transform the  $S_1$  and  $H_2$  solutions of region III(b) into the  $F_2$  functions of region I.

### - The $\nu_1 = \nu_2 = \nu_3 = 1$ limit: $I_3^D(1, 1, 1; Q_1^2, 0, 0, M_1^2, M_2^2, 0)$

All the groups give the correct answer when all the propagators are set equal to one.

As an example, we consider region II(b), so that we can explicitly show the cancellation of the  $\delta$  poles. We fix  $\nu_1 = \nu_3 = 1$ ,  $\nu_2 = 1 + \delta$  and  $D = 4 - 2\epsilon$ . For these choices of the parameters, the hypergeometric functions simplify using the identities given in Sec. A.3.2, and we find

$$I_3^{\{m_1, m_2\}} = -\frac{N_\epsilon}{\epsilon^2} \frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} \left( \frac{Q_1^2}{Q_1^2 - M_2^2} \right)^{2\epsilon} {}_2F_1 \left( 1, 2\epsilon, 1+\epsilon, \frac{M_1^2}{M_2^2 - Q_1^2} \right) \quad (3.28)$$

$$I_3^{\{m_2, p_2\}} = \frac{N_\epsilon}{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-\epsilon-\delta)} \frac{\Gamma(1-\delta)}{\delta} \frac{(Q_1^2)^{2\epsilon}}{(-M_1^2)^\epsilon} (Q_1^2 + M_1^2 - M_2^2)^{-\epsilon-\delta} \quad (3.29)$$

$$I_3^{\{m_1, p_3\}} = \frac{N_\epsilon}{\epsilon^2} \frac{M_2^2}{M_2^2 - M_1^2} \left( \frac{Q_1^2}{-M_2^2} \right)^\epsilon F_2 \left( 1, 1, \epsilon, \epsilon + 1, 1 - \epsilon, \frac{M_1^2}{M_1^2 - M_2^2}, \frac{M_2^2}{Q_1^2} \right) \quad (3.30)$$

$$I_3^{\{p_2, p_3\}} = -\frac{N_\epsilon}{\epsilon} \frac{(-1)^{-\delta}}{\delta (M_2^2 - M_1^2)^\delta} \left( \frac{Q_1^2}{-M_1^2} \right)^\epsilon {}_2F_1 \left( 1, \epsilon, 1 - \delta, \frac{M_2^2 - M_1^2}{Q_1^2} \right), \quad (3.31)$$

where we have defined

$$N_\epsilon = \Gamma(1 + \epsilon) (-Q_1^2)^{-1-\epsilon}. \quad (3.32)$$

Using Eq. (A.50), we can rewrite  $I_3^{\{p_2, p_3\}}$  in the following way

$$\begin{aligned} I_3^{\{p_2, p_3\}} = & -\frac{N_\epsilon}{\epsilon^2} \left( \frac{Q_1^2}{-M_1^2} \right)^\epsilon \frac{Q_1^2}{M_2^2 - M_1^2} {}_2F_1 \left( 1, 1, 1 + \epsilon, \frac{Q_1^2 + M_1^2 - M_2^2}{M_1^2 - M_2^2} \right) \\ & - \frac{N_\epsilon}{(\delta + \epsilon)} \frac{\Gamma(1 + \epsilon + \delta)}{\Gamma(1 + \epsilon)} \frac{(-1)^{-\delta} \Gamma(1 - \delta)}{\delta} \frac{(Q_1^2)^{2\epsilon}}{(-M_1^2)^\epsilon} (Q_1^2 + M_1^2 - M_2^2)^{-\epsilon - \delta}, \end{aligned} \quad (3.33)$$

where we have safely put  $\delta = 0$  in the first line, since this is a finite quantity in  $\delta$ .

We see that the poles in  $\delta$  clearly cancel between  $I_3^{\{m_2, p_2\}}$  and  $I_3^{\{p_2, p_3\}}$  (see Eqs. (3.29) and (3.33)), leaving a finite remainder that is straightforwardly obtained by Taylor expansion about  $\delta = 0$ .

Up to now, we have not required  $\epsilon$  to be small and expressions (3.28)–(3.33) are valid in arbitrary dimension  $D$ .

If we make the usual expansion for  $\epsilon \rightarrow 0$  we recover the result

$$I_3^D(1, 1, 1; Q_1^2, 0, 0, M_1^2, M_2^2, 0) = N_\epsilon \left[ \text{Li}_2 \left( \frac{Q_1^2 + M_1^2 - M_2^2}{M_1^2} \right) - \text{Li}_2 \left( 1 - \frac{M_2^2}{M_1^2} \right) \right] + \mathcal{O}(\epsilon), \quad (3.34)$$

where we have used a series expansion for the integral representation of the functions  ${}_2F_1$  and  $F_2$ , given in Eqs. (A.26) and (A.29), and where  $Q_1^2 \rightarrow Q_1^2 + i0$ , to recover the correct prescription in the Feynman integrals. We describe the details of the  $\epsilon$  expansion in Appendix A.2.1. Expression (3.34) is finite in  $\epsilon$ , as it should be, having no soft or collinear singularities, despite the fact that the individual contributions contain poles in  $\epsilon$ .

- **The  $\nu_1 \rightarrow 0$  limit:**  $I_3^D(0, \nu_2, \nu_3; Q_1^2, 0, 0, M_1^2, M_2^2, 0)$

If we set  $\nu_1 \rightarrow 0$ , we produce a one-mass ( $M_2$ ) bubble integral with external scale  $Q_1^2$  and internal propagators raised to the powers  $\nu_2$  and  $\nu_3$ .

In the different regions (3.17) and for the different groups of Eqs. (3.23)–(3.27), we

have

$$\begin{aligned}
\text{I :} \quad & M_2^2 > Q_1^2 \text{ and } M_2^2 > M_1^2 \implies I_3^D \Big|_{\nu_1=0} = I_3^{\{m_1, q_1\}} \Big|_{\nu_1=0} \\
\text{II(a) :} \quad & Q_1^2 > M_1^2 > M_2^2 \implies I_3^D \Big|_{\nu_1=0} = I_3^{\{m_1, m_2\}} \Big|_{\nu_1=0} + I_3^{\{p_1, p_3\}} \Big|_{\nu_1=0} \\
\text{II(b) :} \quad & Q_1^2 > M_2^2 > M_1^2 \implies I_3^D \Big|_{\nu_1=0} = I_3^{\{m_1, m_2\}} \Big|_{\nu_1=0} + I_3^{\{m_1, p_3\}} \Big|_{\nu_1=0} \\
\text{III(a) :} \quad & M_1^2 > Q_1^2 > M_2^2 \implies I_3^D \Big|_{\nu_1=0} = I_3^{\{m_2, p_1\}} \Big|_{\nu_1=0} + I_3^{\{p_1, p_3\}} \Big|_{\nu_1=0} \\
\text{III(b) :} \quad & M_1^2 > M_2^2 > Q_1^2 \implies I_3^D \Big|_{\nu_1=0} = I_3^{\{p_1, q_1\}} \Big|_{\nu_1=0},
\end{aligned} \tag{3.35}$$

where we have used the shorthand notation

$$I_3^D \Big|_{\nu_1=0} = I_3^D (0, \nu_2, \nu_3; Q_1^2, 0, 0, M_1^2, M_2^2, 0), \tag{3.36}$$

and where the missing terms have been killed by the  $\Gamma(0)$  in the denominator.

It is straightforward to evaluate the different solutions when  $\nu_1 = 0$ . In fact, taking  $I_3^{\{m_1, q_1\}}$  as example, we can use the reduction formula

$$F_2(\alpha, 0, \beta', \gamma, \gamma', x, y) = {}_2F_1(\alpha, \beta', \gamma', y), \tag{3.37}$$

to recover

$$I_3^D(0, \nu_2, \nu_3; Q_1^2, 0, 0, M_1^2, M_2^2, 0) = I_2^D(\nu_2, \nu_3; Q_1^2, M_2^2, 0) \tag{3.38}$$

in region I, that is Eq. (2.52). The same thing happens to the solution in region III(b).

The other part of the bubble integral valid when  $Q_1^2 > M_2^2$  is produced by the other solutions in the region II(a), II(b) and III(a), and agrees with Eq. (2.51).

- **The  $\nu_3 \rightarrow 0$  limit:**  $I_3^D(\nu_1, \nu_2, 0; Q_1^2, 0, 0, M_1^2, M_2^2, 0)$

Likewise, we can set  $\nu_3 \rightarrow 0$  producing a two-mass bubble ( $M_1$  and  $M_2$ ) with external scale  $Q_1^2 = 0$ , for which the result is given in Eq. (2.53). We could repeat the reasoning made for the previous case, and build a table of surviving solutions, analogous to (3.35).

For example, the two terms in Eq. (3.23) collapse to form the correct Gauss' hypergeometric functions when  $M_2^2 > M_1^2$  and  $M_2^2 > Q_1^2$ . Similarly, the result when  $Q_1^2 > M_2^2 > M_1^2$  is produced by the third and fourth term of Eq. (3.25), for region II(b).

- **The  $M_1 \rightarrow 0$  limit:**  $I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, 0, M_2^2, 0)$

Here we expect to reproduce the result for the triangle integral given in Eqs. (3.15) and (3.16), with the exchanges  $M_3 \leftrightarrow M_2$  and  $\nu_3 \leftrightarrow \nu_2$ . Clearly in regions II(a), III(a) and III(b), it is inappropriate to take this limit, since  $M_1^2 > M_2^2$ . In fact, if we just go ahead and apply the limit blindly to the solutions for regions III(a) and III(b), we just

obtain zero.

On the other hand, in regions I and II(b) it does make sense to send  $M_1 \rightarrow 0$  since  $M_1$  is allowed to be the smallest scale present:

$$\begin{aligned} \text{I:} \quad M_2^2 > Q_1^2 \text{ and } M_2^2 > M_1^2 &\implies I_3^D|_{M_1=0} = I_3^{\{m_1, q_1\}}|_{M_1=0} \\ \text{II(b):} \quad Q_1^2 > M_2^2 > M_1^2 &\implies I_3^D|_{M_1=0} = I_3^{\{m_1, m_2\}}|_{M_1=0} + I_3^{\{m_1, p_3\}}|_{M_1=0}, \end{aligned} \quad (3.39)$$

with the shorthand notation

$$I_3^D|_{M_1=0} = I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, 0, M_2^2, 0). \quad (3.40)$$

Again the hypergeometric functions collapse to Gauss'  ${}_2F_1$  functions (see Eqs. (A.17), (A.11) and (A.23))

$$\begin{aligned} F_2(\alpha, \beta, \beta', \gamma, \gamma', 0, y) &= {}_2F_1(\alpha, \beta', \gamma', y), \\ S_1(\alpha, \alpha', \beta, \gamma, \delta, 0, y) &= {}_2F_1(\alpha, \alpha', \gamma, y), \\ H_2(\alpha, \beta, \gamma, \delta, \epsilon, 0, y) &= {}_2F_1(\gamma, \delta, 1 - \alpha, -y), \end{aligned}$$

and we recover, in region I, the result of Eq. (3.16), and in region II(b), the expected result (3.15) for  $M_2^2 < Q_1^2$ .

- **The  $M_2 \rightarrow 0$  limit:  $I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, M_1^2, 0, 0)$**

Taking the limit  $M_2 \rightarrow 0$  provides us with the integral relevant for the exchange of a heavy particle in the decay into two light particles.

As usual, we could merely return to the system and, by setting  $m_2 = 0$ , solve it afresh: there are now  $(m + q + n) = 5$  variables and still  $(n + 1) = 4$  constraints leaving five single-sum solutions. However, it is simpler to take the  $M_2 \rightarrow 0$  limit in the appropriate regions: II(a) and III(a), as can be seen from Eq. (3.17).

We then obtain

$$\begin{aligned} \text{II(a):} \quad Q_1^2 > M_1^2 > M_2^2 &\implies I_3^D|_{M_2=0} = I_3^{\{m_1, m_2\}} + I_3^{\{m_2, p_2\}} + I_3^{\{m_2, p_3\}}|_{M_2=0} \\ \text{III(a):} \quad M_1^2 > Q_1^2 > M_2^2 &\implies I_3^D|_{M_2=0} = I_3^{\{m_2, p_1\}} + I_3^{\{m_2, q_1\}}|_{M_2=0}, \end{aligned} \quad (3.41)$$

where

$$I_3^D|_{M_2=0} = I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, M_1^2, 0, 0). \quad (3.42)$$

The hypergeometric functions collapse to  ${}_3F_2$  functions, according to Eqs. (A.10) and (A.13)

$$\begin{aligned} S_1(\alpha, \alpha', \beta, \gamma, \delta, x, 0) &= {}_3F_2(\alpha, \alpha', \beta, \gamma, \delta, x), \\ S_2(\alpha, \alpha', \beta, \beta', \gamma, 0, y) &= {}_3F_2(1 - \gamma, \beta, \beta', 1 - \alpha, 1 - \alpha', -y), \end{aligned}$$

and we obtain:

if  $Q_1^2 > M_1^2$ , region II(a)

$$\begin{aligned}
I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, M_1^2, 0, 0) &= I_3^{\{m_1\}} + I_3^{\{p_2\}} + I_3^{\{p_3\}} \\
&= (-1)^{\frac{D}{2}} (Q_1^2)^{\frac{D}{2}-\nu_1-\nu_2-\nu_3} \frac{\Gamma(\nu_1 + \nu_2 + \nu_3 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_1 - \nu_2) \Gamma(\frac{D}{2} - \nu_1 - \nu_3)}{\Gamma(\nu_2) \Gamma(\nu_3) \Gamma(D - \nu_1 - \nu_2 - \nu_3)} \\
&\quad \times {}_3F_2\left(\nu_1, 1 + \sigma - D, \sigma - \frac{D}{2}, 1 + \nu_1 + \nu_2 - \frac{D}{2}, 1 + \nu_1 + \nu_3 - \frac{D}{2}, -\frac{M_1^2}{Q_1^2}\right) \\
&+ (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_2} (-M_1^2)^{\frac{D}{2}-\nu_1-\nu_3} \frac{\Gamma(\nu_3 - \nu_2) \Gamma(\frac{D}{2} - \nu_3) \Gamma(\nu_1 + \nu_3 - \frac{D}{2})}{\Gamma(\nu_1) \Gamma(\nu_3) \Gamma(\frac{D}{2} - \nu_2)} \\
&\quad \times {}_3F_2\left(\nu_2, \frac{D}{2} - \nu_3, 1 + \nu_2 - \frac{D}{2}, 1 + \nu_2 - \nu_3, 1 + \frac{D}{2} - \nu_1 - \nu_3, -\frac{M_1^2}{Q_1^2}\right) \\
&+ (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_3} (-M_1^2)^{\frac{D}{2}-\nu_1-\nu_2} \frac{\Gamma(\nu_2 - \nu_3) \Gamma(\frac{D}{2} - \nu_2) \Gamma(\nu_1 + \nu_2 - \frac{D}{2})}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(\frac{D}{2} - \nu_3)} \\
&\quad \times {}_3F_2\left(\nu_3, \frac{D}{2} - \nu_2, 1 + \nu_3 - \frac{D}{2}, 1 + \nu_3 - \nu_2, 1 + \frac{D}{2} - \nu_1 - \nu_2, -\frac{M_1^2}{Q_1^2}\right), \quad (3.43)
\end{aligned}$$

if  $M_1^2 > Q_1^2$ , region III(a)

$$\begin{aligned}
I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, M_1^2, 0, 0) &= I_3^{\{q_1\}} + I_3^{\{p_1\}} \\
&= (-1)^{\frac{D}{2}} (-M_1^2)^{\frac{D}{2}-\nu_1-\nu_2-\nu_3} \frac{\Gamma(\nu_1 + \nu_2 + \nu_3 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_2 - \nu_3)}{\Gamma(\nu_1) \Gamma(\frac{D}{2})} \\
&\quad \times {}_3F_2\left(\nu_2, \nu_3, \nu_1 + \nu_2 + \nu_3 - \frac{D}{2}, 1 + \nu_3 + \nu_2 - \frac{D}{2}, \frac{D}{2}, -\frac{Q_1^2}{M_1^2}\right) \\
&+ (-1)^{\frac{D}{2}} (Q_1^2)^{\frac{D}{2}-\nu_2-\nu_3} (-M_1^2)^{-\nu_1} \frac{\Gamma(\nu_2 + \nu_3 - \frac{D}{2}) \Gamma(\frac{D}{2} - \nu_2) \Gamma(\frac{D}{2} - \nu_3)}{\Gamma(\nu_2) \Gamma(\nu_3) \Gamma(D - \nu_2 - \nu_3)} \\
&\quad \times {}_3F_2\left(\nu_1, \frac{D}{2} - \nu_2, \frac{D}{2} - \nu_3, D - \nu_2 - \nu_3, 1 + \frac{D}{2} - \nu_2 - \nu_3, -\frac{Q_1^2}{M_1^2}\right). \quad (3.44)
\end{aligned}$$

We can further check these results by setting one of the  $\nu_i \rightarrow 0$  to form bubble integrals or by taking one of the limits  $M_1 \rightarrow 0$  or  $Q_1^2 \rightarrow 0$ . In each case, we recover the correct results presented in the earlier sections

### Discussion of system of constraints

As we have already anticipated in Sec. 2.4.2, the homogeneous counterpart of the system of

constraints (3.3), with  $q_2 = q_3 = m_3 = 0$ , is not uniquely solvable. We can build the table of signs for  $p_i$ ,  $q_i$  and  $m_i$ :

$> 0$	$< 0$	uncertain
$p_2, q_1, m_1$	$p_1, p_3, m_2$	
$p_2, p_3, m_1$	$p_1, q_1$	$m_2$
$p_1, q_1$	$p_2, p_3, m_1$	$m_2$
$p_1, p_3, m_2$	$p_2, q_1, m_1$	

but this does not determine the complete list of groups, even if some of them (first and second row, corresponding to region I and region II(b)) are correctly predicted by the table (if we arbitrarily assume that  $m_2$  in the second row has a positive sign).

### 3.3 Equal-mass propagators: $M_1 = M_2 = M_3 = M$

In the special case of  $M_1 = M_2 = M_3 = M$  we use the modified form given in Sec. 2.5. For the most general case of unequal off-shell external legs, there are  $(n + q + 1) = 7$  summation variables and  $(n + 1) = 4$  constraints. The template solution is easily obtained from Eq. (2.58) and the system of constraints is given by

$$\begin{aligned}
q_2 + q_3 + p_1 &= -\nu_1, \\
q_1 + q_3 + p_2 &= -\nu_2, \\
q_1 + q_2 + p_3 &= -\nu_3, \\
p_1 + p_2 + p_3 + q_1 + q_2 + q_3 &= -\frac{D}{2} + m.
\end{aligned} \tag{3.45}$$

#### 3.3.1 $Q_2^2 = Q_3^2 = 0$

In this case, there are five summation variables ( $p_1, p_2, p_3, q_1$  and  $m$ ), and four constraints, and the system admits four solutions out of the possible five.

Based on two-particle cuts of the diagram, we see that there is a threshold at  $Q_1^2 = 4M^2$  corresponding to producing propagators 2 and 3 on-shell. Of the four solutions, three converge when  $Q_1^2 > 4M^2$ , while the remaining solution converges when  $Q_1^2 < 4M^2$ . Recalling  $\sigma = \nu_1 + \nu_2 + \nu_3$  (see Eq. (2.6)), we have:

if  $Q_1^2 > 4M^2$

$$I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, M^2, M^2, M^2) = I_3^{\{m\}} + I_3^{\{p_2\}} + I_3^{\{p_3\}}$$

$$\begin{aligned}
&= (-1)^{\frac{D}{2}} (Q_1^2)^{\frac{D}{2}-\sigma} \frac{\Gamma\left(\sigma - \frac{D}{2}\right) \Gamma\left(\frac{D}{2} - \nu_1 - \nu_2\right) \Gamma\left(\frac{D}{2} - \nu_1 - \nu_3\right)}{\Gamma(\nu_2) \Gamma(\nu_3) \Gamma(D - \sigma)} \\
&\times {}_3F_2\left(\sigma - \frac{D}{2}, 1 + \frac{1}{2}(\sigma - D), \frac{1}{2}(1 + \sigma - D), 1 + \nu_1 + \nu_2 - \frac{D}{2}, 1 + \nu_1 + \nu_3 - \frac{D}{2}, \frac{4M^2}{Q_1^2}\right) \\
&+ (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_2} (-M^2)^{\frac{D}{2}-\nu_1-\nu_3} \frac{\Gamma(\nu_3 - \nu_2) \Gamma\left(\nu_1 + \nu_3 - \frac{D}{2}\right)}{\Gamma(\nu_3) \Gamma(\nu_1 - \nu_2 + \nu_3)} \\
&\times {}_3F_2\left(\nu_2, 1 + \frac{1}{2}(\nu_2 - \nu_1 - \nu_3), \frac{1}{2}(1 - \nu_1 + \nu_2 - \nu_3), 1 + \nu_2 - \nu_3, 1 + \frac{D}{2} - \nu_1 - \nu_3, \frac{4M^2}{Q_1^2}\right) \\
&+ (-1)^{\frac{D}{2}} (Q_1^2)^{-\nu_3} (-M^2)^{\frac{D}{2}-\nu_1-\nu_2} \frac{\Gamma(\nu_2 - \nu_3) \Gamma\left(\nu_1 + \nu_2 - \frac{D}{2}\right)}{\Gamma(\nu_2) \Gamma(\nu_1 + \nu_2 - \nu_3)} \\
&\times {}_3F_2\left(\nu_3, 1 + \frac{1}{2}(\nu_3 - \nu_1 - \nu_2), \frac{1}{2}(1 - \nu_1 + \nu_3 - \nu_2), 1 + \nu_3 - \nu_2, 1 + \frac{D}{2} - \nu_1 - \nu_2, \frac{4M^2}{Q_1^2}\right), \tag{3.46}
\end{aligned}$$

if  $Q_1^2 < 4M^2$

$$\begin{aligned}
I_3^D(\nu_1, \nu_2, \nu_3; Q_1^2, 0, 0, M^2, M^2, M^2) &= I_3^{\{q_1\}} \\
&= (-1)^{\frac{D}{2}} (-M^2)^{\frac{D}{2}-\sigma} \frac{\Gamma\left(\sigma - \frac{D}{2}\right)}{\Gamma(\sigma)} {}_3F_2\left(\nu_2, \nu_3, \sigma - \frac{D}{2}, \frac{\sigma}{2}, \frac{1+\sigma}{2}, \frac{Q_1^2}{4M^2}\right), \tag{3.47}
\end{aligned}$$

where we have made use of formula (2.64). This latter result agrees with that obtained by taking the limit  $Q_2^2 \rightarrow 0$ ,  $Q_3^2 \rightarrow 0$  in the general result given by Boos and Davydychev [13] for a triangle loop integral with three off-shell legs and a single mass  $M$  running round the loop. Equation (3.46) appears to be a new result.

We note that, taking the limit  $\nu_1 \rightarrow 0$  in Eqs (3.46) and (3.47), we reproduce the expected equal-mass bubble integral of Eqs. (2.62) and (2.63)

$$I_3^D(0, \nu_2, \nu_3; Q_1^2, 0, 0, M^2, M^2, M^2) = I_2^D(\nu_2, \nu_3; Q_1^2, M^2, M^2), \tag{3.48}$$

while taking  $M \rightarrow 0$ , only the first term in Eq. (3.46) survives, yielding Eq. (3.11).

However we observe that there are dangerous  $\Gamma$  functions in the second and third lines when  $\nu_2 = \nu_3$ . Therefore to evaluate the integral when  $\nu_2 = \nu_3 = \nu$ , we introduce an additional regulator  $\delta$  such that  $\nu_2 = \nu + \delta$  and  $\nu_3 = \nu$ . As in the previous section, because the result does not depend on  $\delta$ , the limit  $\delta \rightarrow 0$  can be safely taken after the singularities have been canceled.



## 4 Conclusions

Finally let us summarize what we have accomplished in this paper. Changing the number of dimensions  $D$  to evaluate loop integrals is well established and relies on the analytic properties of loop integrals. We have used this property to extend the number of dimensions to negative values as suggested by Halliday and Ricotta. As discussed at length in Sec. 2.2, treating  $D$  as a negative even integer allows a multinomial expansion of the integrand in intermediate steps and, by expanding before and after loop integration we can identify the loop integral as an infinite series, together with constraints on the summation variables. The number of summation parameters is equal to the number of legs  $n$  plus the number of energy scales  $(m + q)$  in the loop, while there are  $n + 1$  constraints, which can be read off from the Feynman graph. The form of the series is specified for arbitrary one-loop integrals and forms a template series into which specific solutions of the system of constraints are inserted. In this way, integration over the parameters is replaced with infinite sums. In each case we immediately identify generalised hypergeometric functions and show how to assemble the complete result valid in a particular kinematic region by considering the convergence properties of the hypergeometric functions. The procedure is as follows:

1. Write down  $\mathcal{P}$ ,  $\mathcal{M}$  and  $\mathcal{Q}$  of Eqs. (2.9), (2.10) and (2.11) by inspection of the graph.
2. Write down the template solution for the particular process. In other words, copy out Eq. (2.23) inserting the correct mass scales and number of terms in  $\mathcal{P}$ ,  $\mathcal{M}$  and  $\mathcal{Q}$ .
3. Construct the system of constraints by counting powers of  $x_i$  in Eqs. (2.20) and (2.21).
4. Solve the system of constraints and insert each solution into the template solution, one at a time.
5. Construct Pochhammer symbols and identify the generalised hypergeometric function.
6. Flip all of the  $\Gamma$  functions in prefactor according to Eq. (2.33).
7. Group the solutions according to their regions of convergence.
8. Evaluate the hypergeometric functions for the specific parameters of interest.

Steps 1-6 are very straightforward and easily achieved with a computer program. Step 7 requires a little more thought, though for the cases we have studied here the convergence regions were easy to identify. To make the procedure useful for phenomenological studies, it is necessary to evaluate the hypergeometric functions for specific values of the parameters. In particular, an integral representation is required. The mathematical literature for Euler integral representations of hypergeometric functions with more than 2 or 3 variables

is quite sparse, and it may be necessary to select a complex integral representation for more complicated functions.

More interesting is the application of NDIM to integrals with more than one loop. Suzuki and Schmidt [23, 24] have made some steps in this direction, although the integrals they have considered are largely of the one-loop insertion type. Given that quite powerful results for one-loop integrals are achieved so easily and generally, we expect that NDIM can play a role in simplifying the task of calculating two- (or more) loop integrals that are necessary to make more precise perturbative predictions within the Standard Model.

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## A Hypergeometric definitions and identities

The purpose of this appendix is to give sufficient information to evaluate the general loop integrals presented in Secs. 2.2 and 3. In Sec. A.1 we give the definitions of the hypergeometric functions as a series together with their regions of convergence. Integral representations are provided in Sec. A.2 together with a description of how to evaluate the integrals in the general case. For specific choices of the  $\nu_i$  the general hypergeometric functions often simplify and some useful identities and analytic-continuation formulae are collected in Sec. A.3.

### A.1 Series representations

The hypergeometric functions of one variable are sums of Pochhammer symbols over a single summation parameter  $m$ , like, for example,

$${}_2F_1(\alpha, \beta, \gamma, x) = \sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta, m)}{(\gamma, m)} \frac{x^m}{m!} \quad (\text{A.1})$$

$${}_3F_2(\alpha, \beta, \beta', \gamma, \gamma', x) = \sum_{m=0}^{\infty} \frac{(\alpha, m)(\beta, m)(\beta', m)}{(\gamma, m)(\gamma', m)} \frac{x^m}{m!}, \quad (\text{A.2})$$

which are convergent when  $|x| < 1$ .

We also meet hypergeometric functions of two variables which can be written as sums over the integers  $m$  and  $n$ :  $F_i$ ,  $i = 1, \dots, 4$  are the Appell functions,  $H_2$  a Horn function and  $S_1$  and  $S_2$  generalised Kampé de Fériet functions:

$$F_1(\alpha, \beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)} \frac{x^m}{m!} \frac{y^n}{n!} \quad (\text{A.3})$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m)(\gamma', n)} \frac{x^m}{m!} \frac{y^n}{n!} \quad (\text{A.4})$$

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m)(\alpha', n)(\beta, m)(\beta', n)}{(\gamma, m+n)} \frac{x^m}{m!} \frac{y^n}{n!} \quad (\text{A.5})$$

$$F_4(\alpha, \beta, \gamma, \gamma', x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\beta, m+n)}{(\gamma, m)(\gamma', n)} \frac{x^m}{m!} \frac{y^n}{n!} \quad (\text{A.6})$$

$$H_2(\alpha, \beta, \gamma, \gamma', \delta, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m-n)(\beta, m)(\gamma, n)(\gamma', n)}{(\delta, m)} \frac{x^m}{m!} \frac{y^n}{n!} \quad (\text{A.7})$$

$$S_1(\alpha, \alpha', \beta, \gamma, \delta, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m+n)(\alpha', m+n)(\beta, m)}{(\gamma, m+n)(\delta, m)} \frac{x^m}{m!} \frac{y^n}{n!} \quad (\text{A.8})$$

$$S_2(\alpha, \alpha', \beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m-n)(\alpha', m-n)(\beta, n)(\beta', n)}{(\gamma, m-n)} \frac{x^m}{m!} \frac{y^n}{n!}. \quad (\text{A.9})$$

These series converge according to the criteria collected in Table 1, The domain of con-

Function	Convergence criteria
$F_1, F_3$	$ x  < 1,  y  < 1$
$F_2, S_1$	$ x  +  y  < 1$
$F_4$	$\sqrt{ x } + \sqrt{ y } < 1$
$H_2, S_2$	$- x  + 1/ y  > 1,  x  < 1,  y  < 1$

Table 1: Convergence regions for some hypergeometric functions of two variables.

vergence of the Appell and Horn functions are well known. That one for  $S_1$  and  $S_2$  may be worked out using Horns general theory of convergence [28].

When one of the arguments vanishes, then the hypergeometric function collapses in a straightforward way. For example, if  $y = 0$  in Eq. (A.8), then only the first term of the series in  $n$  contributes and we are left with the relation,

$$S_1(\alpha, \alpha', \beta, \gamma, \delta, x, 0) = {}_3F_2(\alpha, \alpha', \beta, \gamma, \delta, x). \quad (\text{A.10})$$

Similarly, we have

$$S_1(\alpha, \alpha', \beta, \gamma, \delta, 0, y) = {}_2F_1(\alpha, \alpha', \gamma, y) \quad (\text{A.11})$$

$$S_2(\alpha, \alpha', \beta, \gamma', \gamma, x, 0) = {}_2F_1(\alpha, \alpha', \gamma, x) \quad (\text{A.12})$$

$$S_2(\alpha, \alpha', \beta, \beta', \gamma, 0, y) = {}_3F_2(1 - \gamma, \beta, \beta', 1 - \alpha, 1 - \alpha', -y) \quad (\text{A.13})$$

$$F_1(\alpha, \beta, \beta', \gamma, x, 0) = {}_2F_1(\alpha, \beta, \gamma, x) \quad (\text{A.14})$$

$$F_1(\alpha, \beta, \beta', \gamma, 0, y) = {}_2F_1(\alpha, \beta', \gamma, y) \quad (\text{A.15})$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', x, 0) = {}_2F_1(\alpha, \beta, \gamma, x) \quad (\text{A.16})$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', 0, y) = {}_2F_1(\alpha, \beta', \gamma', y) \quad (\text{A.17})$$

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, 0) = {}_2F_1(\alpha, \beta, \gamma, x) \quad (\text{A.18})$$

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, 0, y) = {}_2F_1(\alpha', \beta', \gamma, y) \quad (\text{A.19})$$

$$F_4(\alpha, \beta, \gamma, \gamma', x, 0) = {}_2F_1(\alpha, \beta, \gamma, x) \quad (\text{A.20})$$

$$F_4(\alpha, \beta, \gamma, \gamma', 0, y) = {}_2F_1(\alpha, \beta, \gamma', y) \quad (\text{A.21})$$

$$H_2(\alpha, \beta, \gamma, \gamma', \delta, x, 0) = {}_2F_1(\alpha, \beta, \delta, x) \quad (\text{A.22})$$

$$H_2(\alpha, \beta, \gamma, \gamma', \delta, 0, y) = {}_2F_1(\gamma, \gamma', 1 - \alpha, -y). \quad (\text{A.23})$$

When one of the parameters vanishes producing a Pochhammer  $(0, n)$ , then the series in  $n$  also terminates. If we have  $(0, m + n)$  then both series terminate. For example

$$F_2(\alpha, 0, \beta', \gamma, \gamma', x, y) = {}_2F_1(\alpha, \beta', \gamma', y) \quad (\text{A.24})$$

$$F_1(0, \beta, \beta', \gamma, x, y) = 1. \quad (\text{A.25})$$

## A.2 Integral representations

Euler integral representations of the hypergeometric series of one and two variables are well known [29, 30] and we list them here. We know of no integral representation for the  $H_2$  function and for the closely related  $S_2$  function.

$${}_2F_1(\alpha, \beta, \gamma, x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \times \int_0^1 du u^{\beta-1} (1-u)^{\gamma-\beta-1} (1-ux)^{-\alpha}$$

$$\text{Re}(\beta) > 0, \quad \text{Re}(\gamma - \beta) > 0. \quad (\text{A.26})$$

$${}_3F_2(\alpha, \beta, \gamma, \delta, \epsilon, x) = \frac{\Gamma(\delta)\Gamma(\epsilon)}{\Gamma(\beta)\Gamma(\delta - \beta)\Gamma(\gamma)\Gamma(\epsilon - \gamma)}$$

$$\times \int_0^1 du \int_0^1 dv u^{\beta-1} v^{\gamma-1} (1-u)^{\delta-\beta-1} (1-v)^{\epsilon-\gamma-1} (1-uvx)^{-\alpha}$$

$$\text{Re}(\beta) > 0, \quad \text{Re}(\delta - \beta) > 0, \quad \text{Re}(\gamma) > 0, \quad \text{Re}(\epsilon - \gamma) > 0. \quad (\text{A.27})$$

$$F_1(\alpha, \beta, \beta', \gamma, x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 du u^{\alpha-1} (1-u)^{\gamma-\alpha-1} (1-ux)^{-\beta} (1-uy)^{-\beta'}$$

$$\operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\gamma - \alpha) > 0. \quad (\text{A.28})$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}$$

$$\times \int_0^1 du \int_0^1 dv u^{\beta-1} v^{\beta'-1} (1-u)^{\gamma-\beta-1} (1-v)^{\gamma'-\beta'-1} (1-ux-vy)^{-\alpha}$$

$$\operatorname{Re}(\beta) > 0, \quad \operatorname{Re}(\beta') > 0, \quad \operatorname{Re}(\gamma - \beta) > 0, \quad \operatorname{Re}(\gamma' - \beta') > 0. \quad (\text{A.29})$$

$$F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}$$

$$\times \int_{u \geq 0, v \geq 0}^{u+v \leq 1} du dv u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-ux)^{-\alpha} (1-vy)^{-\alpha'}$$

$$\operatorname{Re}(\beta) > 0, \quad \operatorname{Re}(\beta') > 0, \quad \operatorname{Re}(\gamma - \beta - \beta') > 0. \quad (\text{A.30})$$

$$F_4(\alpha, \beta, \gamma, \gamma', x(1-y), y(1-x)) = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma'-\beta)}$$

$$\times \int_0^1 du \int_0^1 dv u^{\alpha-1} v^{\beta-1} (1-u)^{\gamma-\alpha-1} (1-v)^{\gamma'-\beta-1}$$

$$\times (1-ux)^{\alpha-\gamma-\gamma'+1} (1-vy)^{\beta-\gamma-\gamma'+1} (1-ux-vy)^{\gamma+\gamma'-\alpha-\beta'-1}$$

$$\operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\beta) > 0, \quad \operatorname{Re}(\gamma - \alpha) > 0, \quad \operatorname{Re}(\gamma' - \beta) > 0. \quad (\text{A.31})$$

$$S_1(\alpha, \alpha', \beta, \gamma, \delta, x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 du u^{\alpha-1} (1-u)^{\gamma-\alpha-1} F_2(\alpha', \beta, 1, \delta, 1, ux, uy)$$

$$= \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)\Gamma(\beta)\Gamma(\delta-\beta)}$$

$$\times \int_0^1 du \int_0^1 dv u^{\alpha-1} v^{\beta-1} (1-u)^{\gamma-\alpha-1} (1-v)^{\delta-\beta-1} (1-uvx-uy)^{-\alpha'}$$

$$\operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\gamma - \alpha) > 0, \quad \operatorname{Re}(\beta) > 0, \quad \operatorname{Re}(\delta - \beta) > 0. \quad (\text{A.32})$$

### A.2.1 Example of explicit evaluation of an integral representation

In working out the integral representation for hypergeometric functions in  $D = 4 - 2\epsilon$  dimensions, we have often to deal with the  $\epsilon$  expansion of integrals of the form

$$I(x) = \int_0^1 du d(u) f(u), \quad (\text{A.33})$$

$$d(u) = u^{-1+\alpha\epsilon}(1-u)^{-1+\beta\epsilon} \quad (\text{A.34})$$

where  $\alpha$  and  $\beta$  are real numbers and  $f(u)$  is a smooth function in the domain  $0 \leq u \leq 1$ : in particular, it is finite at the boundary points.

The procedure to deal with this kind of integrals is quite standard. The integral has a pole in  $\epsilon$  when the integration variable  $u$  approaches either of the end points. We concentrate first on the point  $u = 0$ , and we rewrite the integral in such a way to expose the pole in  $\epsilon$

$$I(x) = \int_0^1 du d(u) f(0) + \int_0^1 du d(u) [f(u) - f(0)] = I_{[1]} + I_{[2]}. \quad (\text{A.35})$$

The integral  $I_{[1]}$  can be easily done

$$I_{[1]} = f(0) \frac{\Gamma(\alpha\epsilon) \Gamma(\beta\epsilon)}{\Gamma((\alpha+\beta)\epsilon)} = \frac{f(0)}{\epsilon} \frac{\alpha + \beta}{\alpha\beta} \frac{\Gamma(1+\alpha\epsilon) \Gamma(1+\beta\epsilon)}{\Gamma(1+(\alpha+\beta)\epsilon)}, \quad (\text{A.36})$$

and the integrand of  $I_{[2]}$  is now finite in the limit  $u \rightarrow 0$ . In fact, we can make a Taylor expansion

$$f(u) - f(0) = uf'(0) + \frac{u^2}{2!} f''(0) + \dots \equiv ug(u), \quad (\text{A.37})$$

and write  $I_{[2]}$  as

$$I_{[2]} = \int_0^1 du d(u) ug(u) = \int_0^1 du u^{\alpha\epsilon}(1-u)^{-1+\beta\epsilon} g(u). \quad (\text{A.38})$$

We repeat now the same steps done for Eq. (A.35) with respect to the point  $u = 1$ , to obtain

$$I_{[2]} = \int_0^1 du u^{\alpha\epsilon}(1-u)^{-1+\beta\epsilon} g(1) + \int_0^1 du u^{\alpha\epsilon}(1-u)^{-1+\beta\epsilon} [g(u) - g(1)] = I_{[3]} + I_{[4]}. \quad (\text{A.39})$$

The integral  $I_{[3]}$  gives

$$I_{[3]} = g(1) \frac{\Gamma(1+\alpha\epsilon) \Gamma(\beta\epsilon)}{\Gamma(1+(\alpha+\beta)\epsilon)} = \frac{f(1) - f(0)}{\beta\epsilon} \frac{\Gamma(1+\alpha\epsilon) \Gamma(1+\beta\epsilon)}{\Gamma(1+(\alpha+\beta)\epsilon)}, \quad (\text{A.40})$$

while  $I_{[4]}$  is finite at  $u \rightarrow 1$

$$I_{[4]} = \int_0^1 du u^{\alpha\epsilon}(1-u)^{\beta\epsilon} h(u), \quad g(u) - g(1) \equiv (1-u)h(u), \quad (\text{A.41})$$

and can be solved with an  $\epsilon$  expansion of the integrand. Adding all the contributions together we have

$$I(x) = \frac{1}{\alpha \beta \epsilon} [\beta f(0) + \alpha f(1)] \frac{\Gamma(1 + \alpha \epsilon) \Gamma(1 + \beta \epsilon)}{\Gamma(1 + (\alpha + \beta) \epsilon)} + \int_0^1 du u^{\alpha \epsilon} (1 - u)^{\beta \epsilon} h(u), \quad (\text{A.42})$$

where

$$h(u) = \frac{1}{u(1 - u)} \left( f(u) - (1 - u) f(0) - u f(1) \right). \quad (\text{A.43})$$

In the case where we have two integration variables, the procedure outlined above can be re-iterated in a straightforward manner. To illustrate the procedure, we evaluate explicitly the  $F_2$  functions of Eq. (3.30) to  $\mathcal{O}(\epsilon^2)$ .

The integral representation for  $F_2$  (see Eq. (A.29)) is given by

$$F_2(1, 1, \epsilon, \epsilon + 1, 1 - \epsilon, x, y) = \frac{\epsilon^2 \Gamma(1 - \epsilon)}{\Gamma(1 + \epsilon) \Gamma(1 - 2\epsilon)} I(x, y), \quad (\text{A.44})$$

where

$$I(x, y) = \int_0^1 du dv d(u, v) f(u, v), \quad (\text{A.45})$$

and

$$\begin{aligned} d(u, v) &= v^{-1+\epsilon} (1 - u)^{-1+\epsilon} (1 - v)^{-2\epsilon} \\ f(u, v) &= (1 - ux - vy)^{-1}, \end{aligned}$$

and  $I(x, y)$  must be computed to  $\mathcal{O}(\epsilon^0)$ . In order to expose the poles (see Eq. (A.35)), we add and subtract the value of the finite function  $f(u, v)$ , computed at the boundary points, in the following way:

$$\begin{aligned} I(x, y) &= \int_0^1 du dv d(u, v) \left\{ [f(1, 0)] + [f(u, 0) - f(1, 0)] + [f(1, v) - f(1, 0)] \right. \\ &\quad \left. + [f(u, v) - f(u, 0) - f(1, v) + f(1, 0)] \right\} \\ &= I_{[1]} + I_{[2]} + I_{[3]} + I_{[4]}. \end{aligned} \quad (\text{A.46})$$

We are now in a position to evaluate the single contributions in the square brackets. In fact

$$\begin{aligned} I_{[1]} &= (1 - x)^{-1} \int_0^1 du (1 - u)^{-1+\epsilon} \int_0^1 dv v^{-1+\epsilon} (1 - v)^{-2\epsilon} = (1 - x)^{-1} \frac{\Gamma(1 + \epsilon) \Gamma(1 - 2\epsilon)}{\epsilon^2 \Gamma(1 - \epsilon)} \\ I_{[2]} &= \frac{-x}{1 - x} \frac{\Gamma(1 + \epsilon) \Gamma(1 - 2\epsilon)}{\epsilon \Gamma(1 - \epsilon)} \int_0^1 du \frac{(1 - u)^\epsilon}{(1 - ux)} \\ I_{[3]} &= \frac{(1 - x)^{-1}}{\epsilon} \int_0^1 dv \frac{v^\epsilon (1 - v)^{-2\epsilon}}{1 - x - vy} \\ I_{[4]} &= \frac{xy}{1 - x} \int_0^1 du dv (1 - u)^\epsilon v^\epsilon (1 - v)^{-2\epsilon} \frac{(vy + ux + x - 2)}{(1 - ux)(1 - x - vy)(1 - vy - ux)}. \end{aligned} \quad (\text{A.47})$$

The remaining integrals are finite in the limit  $\epsilon \rightarrow 0$ , so that we can make a Taylor expansion to  $\mathcal{O}(\epsilon)$  for the integrands of  $I_{[2]}$  and  $I_{[3]}$ , and we can put directly  $\epsilon = 0$  in  $I_{[4]}$ . Recalling the definition of the dilogarithm function

$$\text{Li}_2(x) = - \int_0^x dz \frac{\log(1-z)}{z} \quad x \leq 1, \quad (\text{A.48})$$

it is straightforward to carry on the last integrations and express the result in terms of  $\text{Li}_2$  functions, as done in Eq. (3.34).

### A.3 Identities amongst the hypergeometric functions

There are three kinds of identities that relate hypergeometric functions. First there are analytic continuations which connect functions in different regions of convergence. Second are reduction formula which allow the functions to be expressed as simpler series for certain values of the parameters. Finally there are transformations which relate the same functions with different arguments.

#### A.3.1 Analytic continuation formulae

Here we give only those analytic continuation properties that relate the argument and inverse argument. Gauss' hypergeometric function has the following analytic continuation properties (see for example [30])

$$\begin{aligned} {}_2F_1(\alpha, \beta, \gamma, z) &= (-z)^{-\alpha} \frac{\Gamma(\gamma) \Gamma(\beta - \alpha)}{\Gamma(\beta) \Gamma(\gamma - \alpha)} {}_2F_1\left(\alpha, 1 + \alpha - \gamma, 1 + \alpha - \beta, \frac{1}{z}\right) \\ &\quad + (-z)^{-\beta} \frac{\Gamma(\gamma) \Gamma(\alpha - \beta)}{\Gamma(\alpha) \Gamma(\gamma - \beta)} {}_2F_1\left(\beta, 1 + \beta - \gamma, 1 + \beta - \alpha, \frac{1}{z}\right). \\ |\arg(-z)| &< \pi, \end{aligned} \quad (\text{A.49})$$

$$\begin{aligned} {}_2F_1(\alpha, \beta, \gamma, z) &= z^{-\alpha} \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} {}_2F_1\left(\alpha, 1 + \alpha - \gamma, 1 + \alpha + \beta - \gamma, 1 - \frac{1}{z}\right) \\ &\quad + z^{\alpha - \gamma} (1 - z)^{\gamma - \alpha - \beta} \frac{\Gamma(\gamma) \Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha) \Gamma(\beta)} {}_2F_1\left(\gamma - \alpha, 1 - \alpha, 1 + \gamma - \alpha - \beta, 1 - \frac{1}{z}\right) \\ |\arg(z)| &< \pi, \quad |\arg(1 - z)| < \pi. \end{aligned} \quad (\text{A.50})$$

The corresponding analytic continuation of the hypergeometric functions with two variables are summarised in Table 2. There are many possible analytic continuations; however, we list only those that are relevant to link the groups of solutions for the two-mass triangle



integral discussed in Sec. 3.2, that is the connections between the Appell and Horn functions. The others are easily derived by summing the series with respect to one of the summation variables to obtain an  ${}_2F_1$ , applying Eq. (A.49), rewriting Gauss' hypergeometric function as a series and reidentifying the double series. We see that these functions appear to form a group.

Function	Continued in terms of
$F_4(x, y)$	$F_4(x/y, 1/y), F_4(y/x, 1/x)$
$F_3(x, y)$	$H_2(1/x, -y), H_2(1/y, -x), F_2(1/x, 1/y)$
$H_2(x, y)$	$F_2(x, -1/y)$
$F_2(x, y)$	$S_1(-y/x, 1/x), H_2(y, -1/x), S_1(-x/y, 1/y), H_2(x, -1/y)$
$H_2(x, y)$	$F_3(1/x, -y), S_2(1/x, -xy)$
$S_1(x, y)$	$F_2(-x/y, 1/y)$
$S_2(x, y)$	$H_2(1/x, -xy)$

Table 2: Analytic continuation for the hypergeometric functions of two variables.

$$\begin{aligned}
F_4(\alpha, \beta, \gamma, \gamma', x, y) &= \frac{\Gamma(\gamma') \Gamma(\beta - \alpha)}{\Gamma(\gamma' - \alpha) \Gamma(\beta)} (-y)^{-\alpha} F_4\left(\alpha, \alpha + 1 - \gamma', \gamma, \alpha + 1 - \beta, \frac{x}{y}, \frac{1}{y}\right) \\
&+ \frac{\Gamma(\gamma') \Gamma(\alpha - \beta)}{\Gamma(\gamma' - \beta) \Gamma(\alpha)} (-y)^{-\beta} F_4\left(\beta, \beta + 1 - \gamma', \gamma, \beta + 1 - \alpha, \frac{x}{y}, \frac{1}{y}\right) \quad (\text{A.51})
\end{aligned}$$

$$\begin{aligned}
F_3(\alpha, \alpha', \beta, \beta', \gamma, x, y) &= \frac{\Gamma(\beta - \alpha) \Gamma(\gamma)}{\Gamma(\gamma - \alpha) \Gamma(\beta)} (-x)^{-\alpha} H_2\left(\alpha + 1 - \gamma, \alpha, \alpha', \beta', \alpha + 1 - \beta, \frac{1}{x}, -y\right) \\
&+ \frac{\Gamma(\alpha - \beta) \Gamma(\gamma)}{\Gamma(\gamma - \beta) \Gamma(\alpha)} (-x)^{-\beta} H_2\left(\beta + 1 - \gamma, \beta, \alpha', \beta', \beta + 1 - \alpha, \frac{1}{x}, -y\right) \quad (\text{A.52})
\end{aligned}$$

$$\begin{aligned}
H_2(\alpha, \beta, \gamma, \gamma', \delta, x, y) &= \frac{\Gamma(\gamma' - \gamma) \Gamma(1 - \alpha)}{\Gamma(1 - \alpha - \gamma) \Gamma(\gamma')} (y)^{-\gamma} F_2\left(\alpha + \gamma, \beta, \gamma, \delta, \gamma + 1 - \gamma', x, -\frac{1}{y}\right) \\
&+ \frac{\Gamma(\gamma - \gamma') \Gamma(1 - \alpha)}{\Gamma(1 - \alpha - \gamma') \Gamma(\gamma)} (y)^{-\gamma'} F_2\left(\alpha + \gamma', \beta, \gamma', \delta, \gamma' + 1 - \gamma, x, -\frac{1}{y}\right) \quad (\text{A.53})
\end{aligned}$$

$$\begin{aligned}
F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) &= \frac{\Gamma(\beta - \alpha) \Gamma(\gamma)}{\Gamma(\gamma - \alpha) \Gamma(\beta)} (-x)^{-\alpha} S_1\left(\alpha, \alpha + 1 - \gamma, \beta', \alpha + 1 - \beta, \gamma', -\frac{y}{x}, \frac{1}{x}\right) \\
&+ \frac{\Gamma(\alpha - \beta) \Gamma(\gamma)}{\Gamma(\gamma - \beta) \Gamma(\alpha)} (-x)^{-\beta} H_2\left(\alpha - \beta, \beta', \beta, \beta + 1 - \gamma, \gamma', y, -\frac{1}{x}\right) \quad (\text{A.54})
\end{aligned}$$

$$\begin{aligned}
H_2(\alpha, \beta, \gamma, \gamma', \delta, x, y) &= \frac{\Gamma(\beta - \alpha) \Gamma(\delta)}{\Gamma(\delta - \alpha) \Gamma(\beta)} (-x)^{-\alpha} S_2\left(\alpha, \alpha + 1 - \delta, \gamma', \gamma, \alpha + 1 - \beta, \frac{1}{x}, -xy\right) \\
&+ \frac{\Gamma(\alpha - \beta) \Gamma(\delta)}{\Gamma(\delta - \beta) \Gamma(\alpha)} (-x)^{-\beta} F_3\left(\beta, \gamma', \beta + 1 - \delta, \gamma, \beta + 1 - \alpha, \frac{1}{x}, -y\right)
\end{aligned} \tag{A.55}$$

$$\begin{aligned}
S_1(\alpha, \alpha', \beta, \gamma, \delta, x, y) &= \frac{\Gamma(\alpha' - \alpha) \Gamma(\gamma)}{\Gamma(\gamma - \alpha) \Gamma(\alpha')} (-y)^{-\alpha} F_2\left(\alpha, \beta, \alpha + 1 - \gamma, \delta, \alpha + 1 - \alpha', -\frac{x}{y}, \frac{1}{y}\right) \\
&+ \frac{\Gamma(\alpha - \alpha') \Gamma(\gamma)}{\Gamma(\gamma - \alpha') \Gamma(\alpha)} (-y)^{-\alpha'} F_2\left(\alpha', \beta, \alpha' + 1 - \gamma, \delta, \alpha' + 1 - \alpha, -\frac{x}{y}, \frac{1}{y}\right)
\end{aligned} \tag{A.56}$$

$$\begin{aligned}
S_2(\alpha, \alpha', \beta, \beta', \gamma, x, y) &= \frac{\Gamma(\alpha' - \alpha) \Gamma(\gamma)}{\Gamma(\gamma - \alpha) \Gamma(\alpha')} (-x)^{-\alpha} H_2\left(\alpha, \alpha + 1 - \gamma, \beta', \beta, \alpha + 1 - \alpha', \frac{1}{x}, -xy\right) \\
&+ \frac{\Gamma(\alpha - \alpha') \Gamma(\gamma)}{\Gamma(\gamma - \alpha') \Gamma(\alpha)} (-x)^{-\alpha'} H_2\left(\alpha', \alpha' + 1 - \gamma, \beta', \beta, \alpha' + 1 - \alpha, \frac{1}{x}, -xy\right)
\end{aligned} \tag{A.57}$$

### A.3.2 Reduction formulae

The  $F_4$  functions describing the massive bubble and the off-shell massless triangle have the following reduction formulae which leave a single remaining Euler integral at most [30, 31]

$$\begin{aligned}
F_4\left(\alpha, \beta, \gamma, \beta, -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right) \\
= (1-x)^\alpha (1-y)^\alpha F_1(\alpha, \gamma - \beta, 1 + \alpha - \gamma, \gamma, x, xy), \tag{A.58}
\end{aligned}$$

$$F_4\left(\alpha, \beta, \alpha, \beta, -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right) = (1-xy)^{-1} (1-x)^\beta (1-y)^\alpha, \tag{A.59}$$

$$\begin{aligned}
F_4\left(\alpha, \beta, \beta, \beta, -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right) \\
= (1-x)^\alpha (1-y)^\alpha {}_2F_1(\alpha, 1 + \alpha - \beta, \beta, xy), \tag{A.60}
\end{aligned}$$

$$\begin{aligned}
F_4\left(\alpha, \beta, 1 + \alpha - \beta, \beta, -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right) \\
= (1-y)^\alpha {}_2F_1\left(\alpha, \beta, 1 + \alpha - \beta, -\frac{x(1-y)}{1-x}\right). \tag{A.61}
\end{aligned}$$

Similar reductions for the other functions of two variables are

$$F_1(\alpha, \beta, \beta', \beta + \beta', x, y) = (1-y)^{-\alpha} {}_2F_1\left(\alpha, \beta, \beta + \beta', \frac{x-y}{1-y}\right) \tag{A.62}$$

$$F_2(\alpha, \beta, \beta', \gamma, \alpha, x, y) = (1-y)^{-\beta'} F_1\left(\beta, \alpha - \beta', \beta', \gamma, x, \frac{x}{1-y}\right) \quad (\text{A.63})$$

$$F_2(\alpha, \beta, \beta', \alpha, \gamma', x, y) = (1-x)^{-\beta} F_1\left(\beta', \beta, \alpha - \beta, \gamma', \frac{y}{1-x}, y\right) \quad (\text{A.64})$$

$$F_2(\alpha, \beta, \beta', \beta, \gamma', x, y) = (1-x)^{-\alpha} {}_2F_1\left(\alpha, \beta', \gamma', \frac{y}{1-x}\right) \quad (\text{A.65})$$

$$F_2(\alpha, \beta, \beta', \alpha, \alpha, x, y) = (1-x)^{-\beta} (1-y)^{-\beta'} {}_2F_1\left(\beta, \beta', \alpha, \frac{xy}{(1-x)(1-y)}\right) \quad (\text{A.66})$$

$$F_2(\alpha, \beta, \beta', \alpha, \beta', x, y) = (1-y)^{\beta-\alpha} (1-x-y)^{-\beta} \quad (\text{A.67})$$

$$F_2(\alpha, \beta, \beta', \beta, \beta', x, y) = (1-x-y)^{-\alpha} \quad (\text{A.68})$$

$$F_3(\alpha, \gamma - \alpha, \beta, \beta', \gamma, x, y) = (1-y)^{-\beta'} F_1\left(\alpha, \beta, \beta', \gamma, x, \frac{y}{y-1}\right) \quad (\text{A.69})$$

$$F_3(\alpha, \gamma - \alpha, \beta, \gamma - \beta, \gamma, x, y) = (1-y)^{\alpha+\beta-\gamma} {}_2F_1(\alpha, \beta, \gamma, x+y-xy), \quad (\text{A.70})$$

while for certain values of the parameters the  $H_2$  function reduces to an  $F_2$  or  $F_1$

$$H_2(\alpha, \beta, \gamma, \delta - \alpha, \delta, x, y) = (1-x)^{-\beta} F_2\left(\delta - \alpha, \beta, \gamma, \delta, 1 - \alpha, -\frac{x}{1-x}, -y\right), \quad (\text{A.71})$$

$$H_2(\alpha, \beta, \gamma, 1 - \alpha, \delta, x, y) = (1+y)^{-\gamma} F_1\left(\beta, \alpha, \gamma, \delta, x, \frac{xy}{1+y}\right). \quad (\text{A.72})$$

The  $S_1$  and  $S_2$  functions we have introduced are less well known. From the integral representation or manipulating the series using Eq. (A.76) we find

$$S_1(\alpha, \alpha', \beta, \alpha, \delta, x, y) = (1-y)^{-\alpha'} {}_2F_1\left(\alpha', \beta, \delta, \frac{x}{1-y}\right), \quad (\text{A.73})$$

$$S_1(\alpha, \alpha', \beta, \gamma, \beta, x, y) = {}_2F_1(\alpha, \alpha', \gamma, x+y), \quad (\text{A.74})$$

$$S_2(\alpha, \alpha', \beta, \beta', \alpha, x, y) = (1-x)^{-\alpha'} {}_2F_1(\beta, \beta', 1 - \alpha', -y(1-x)). \quad (\text{A.75})$$

### A.3.3 Transformation formulae

A useful formula connecting Gauss' hypergeometric function to itself is

$$\begin{aligned} {}_2F_1(\alpha, \beta, \gamma, z) &= (1-z)^{-\alpha} {}_2F_1\left(\alpha, \gamma - \beta, \gamma, \frac{z}{z-1}\right) \\ &= (1-z)^{-\beta} {}_2F_1\left(\gamma - \alpha, \beta, \gamma, \frac{z}{z-1}\right) \\ &= (1-z)^{\gamma-\alpha-\beta} {}_2F_1(\gamma - \alpha, \gamma - \beta, \gamma, z). \end{aligned} \quad (\text{A.76})$$

Using the above results, then

$$\begin{aligned}
F_1(\alpha, \beta, \beta', \gamma, x, y) &= (1-x)^{-\beta}(1-y)^{-\beta'} F_1\left(\gamma - \alpha, \beta, \beta', \gamma, \frac{x}{x-1}, \frac{y}{y-1}\right) \\
&= (1-x)^{-\alpha} F_1\left(\alpha, \gamma - \beta - \beta', \beta', \gamma, \frac{x}{x-1}, \frac{x-y}{x-1}\right) \\
&= (1-y)^{-\alpha} F_1\left(\alpha, \beta, \gamma - \beta - \beta', \gamma, \frac{y-x}{y-1}, \frac{y}{y-1}\right) \tag{A.77}
\end{aligned}$$

$$\begin{aligned}
F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) &= (1-x)^{-\alpha} F_2\left(\alpha, \gamma - \beta, \beta', \gamma, \gamma', \frac{x}{x-1}, \frac{y}{1-x}\right) \\
&= (1-y)^{-\alpha} F_2\left(\alpha, \beta, \gamma' - \beta', \gamma, \gamma', \frac{x}{1-y}, \frac{y}{y-1}\right) \\
&= (1-x-y)^{-\alpha} F_2\left(\alpha, \gamma - \beta, \gamma' - \beta', \gamma, \gamma', \frac{x}{x+y-1}, \frac{y}{x+y-1}\right). \tag{A.78}
\end{aligned}$$

## References

- [1] C.G. Bollini and J.J. Giambiagi, *Nuovo Cim.* **B12** (1972) 20;  
J.F. Ashmore, *Lett. Nuovo Cim.* **4** (1972) 289.
- [2] G. 't Hooft and M. Veltman, *Nucl. Phys.* **B44** (1972) 189.
- [3] See for example, J. Collins, “Renormalisation”, Cambridge University Press (1984).
- [4] G. 't Hooft and M. Veltman, *Nucl. Phys.* **B153** (1979) 365.
- [5] K.G. Chetyrkin and F.V. Tkachov, “A new approach to evaluation of multiloop Feynman integrals”, INR Preprint R-0118, Institute of Nuclear research, Moscow (1979).
- [6] A.E. Terrano, *Phys. Lett.* **B93** (1980) 424.
- [7] K.G. Chetyrkin, A.L. Kataev and F.V. Tkachov, *Nucl. Phys.* **B174** (1980) 345;  
K.G. Chetyrkin and F.V. Tkachov, *Nucl. Phys.* **B192** (1981) 159.
- [8] N.I. Ussyukina, *Teor. Mat. Fiz.* **54** (1983) 124;  
V.V. Belokurov and N.I. Ussyukina, *J. Phys.* **A16** (1983) 2811;  
D.I. Kazakov, *Teor. Mat. Fiz.* **58** (1984) 345;  
D.I. Kazakov and A.V. Kotikov, *Teor. Mat. Fiz.* **73** (1987) 348.
- [9] G.J. van Oldenborgh and J.A.M. Vermaseren, *Z. Phys.* **C46** (1990) 425.

- [10] A.V. Kotikov, *Phys. Lett.* **B254** (1991) 158, *Phys. Lett.* **B259** (1991) 314, *Phys. Lett.* **B267** (1991) 123.
- [11] Z. Bern, L. Dixon and D.A. Kosower, *Nucl. Phys.* **B412** (1994) 751.
- [12] G. Kramer and B. Lampe, *J. Math. Phys.* **28** (1987) 945.
- [13] E.E. Boos and A.I. Davydychev, *Theor. Math. Phys.* **89** (1991) 1052; see also A.I. Davydychev, *J. Math. Phys.* **32** (1991) 1052 and *J. Math. Phys.* **33** (1992) 358.
- [14] V.A. Smirnov, “Analytical result for the dimensionally regularized massless on-shell double box”, [hep-ph/9905323](#).
- [15] A.I. Davydychev, Proc. “Quarks-92”, QCD161:A42:1992, World Scientific, Singapore, (1993), [hep-ph/9307323](#).
- [16] M.A. Sanchis-Lozano, “On the connection between generalized hypergeometric functions and dilogarithms”, [hep-ph/9511322](#);  
L.G. Cabral-Rosetti and M.A. Sanchis-Lozano, “Generalized hypergeometric functions and the evaluation of one loop integrals in Feynman diagrams”, [hep-ph/9809213](#).
- [17] S. Bauberger, F.A. Berends, M. Böhm and M. Buza, *Nucl. Phys.* **B434** (1995) 383;  
F.A. Berends, M. Buza, M. Böhm and R. Scharf, *Z. Phys.* **C63** (1994) 227.
- [18] J.M. Campbell, E.W.N. Glover and D.J. Miller, *Nucl. Phys.* **B498** (1997) 397.
- [19] O.V. Tarasov, *Phys. Rev.* **D54** (1996) 6479, *Nucl. Phys.* **B480** (1996) 397.
- [20] O.V. Tarasov, *Acta Phys. Polon.* **B29** (1998) 2655;  
J. Fleisher, F. Jegerlehner and O.V. Tarasov, “Algebraic reduction of one-loop Feynman graph amplitudes”, [hep-ph/9907327](#).
- [21] I.G. Halliday and R.M. Ricotta, *Phys. Lett.* **B193** (1987) 241;  
R.M. Ricotta, ‘Negative dimensions in field theory’, J.J. Giambiagi Festschrift, ed. H. Falomir, p. 350 (1990).
- [22] G.V. Dunne and I.G. Halliday, *Phys. Lett.* **B193** (1987) 247 and *Nucl. Phys.* **B308** (1998) 589.
- [23] A.T. Suzuki and A.G.M. Schmidt, *J. High Energy Phys.* **9709** (1997) 002; *Eur. Phys. J.* **C5** (1998) 175; “Solutions for a massless off-shell two loop three point vertex”, [hep-th/9712104](#); *Phys. Rev.* **D58** (1998) 047701; *J. Phys.* **A31** (1998) 8023.
- [24] A.T. Suzuki and A.G.M. Schmidt, “Negative dimensional approach for scalar two loop three point and three loop two point integrals”, [hep-th/9904195](#).

- [25] A.T. Suzuki and A.G.M. Schmidt, “Feynman integrals with tensorial structure in the negative dimensional integration scheme”, [hep-th/9903076](#).
- [26] A.T. Suzuki and A.G.M. Schmidt, “Negative dimensional integration for massive four point functions. 1. The standard solutions”, [hep-th/9707187](#); “Negative dimensional integration for massive four point functions. II. New solutions”, [hep-th/9709167](#).
- [27] D. Broadhurst, *Phys. Lett.* **197B** (1987) 179.
- [28] H. Exton, “Multiple Hypergeometric Functions and Applications”, Ellis Horwood (1976).
- [29] P. Appell and J. Kampé de Fériet, “Fonctions Hypergéométriques et Hypersphériques, Polynomes D’Hermite”, Gauthiers-Villars, Paris (1926).
- [30] A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, “Higher Transcendental Functions”, Vol. I, McGraw Hill (1953).
- [31] W.N. Bailey, “Generalised Hypergeometric Series”, Cambridge University Press (1966).